Real Energy Dark Matter Particles with Mostly complex Limiting Velocities in Either Quadratic or Linear Forms

Josip Šoln

ABSTRACT

New forms of matter, such as dark matter, seek new mathematical approaches. Here we describe an approach through the bicubic equation of limiting particle velocity formalism. The bicubic equation discriminant D in this undertaking satisfy $D \geq 0$ determined by the congruent parameter z satisfying $z^2 \geq 1$, where formally $z(m) = 3\sqrt{3}mv^2/2E$, with m , v , and E being respectively, particle mass, velocity and energy. Also, nonlinearly related to the particle congruent parameter z is the particle congruent angle α . These two dimensionless parameters z and α simplify expressions in the bicubic equation limiting particle velocity formalism when evaluating, in both quadratic and linear forms, the three particle limiting velocities, c_1 , c_2 and c_3 ; (primary, secondary and tertiary) in terms of the ordinary particle velocity, v , in either quadratic or linear forms. Corresponding to these limiting velocities, for fixed values of congruent parameter $z(m)$ and congruent angle $\alpha(m)$, one then deduces, with equal values, dark matter particle energies $E(c_1)$, $E(c_2)$ and $E(c_3)$ naturally in terms of limiting velocity squares c_1^2 , c_2^2 and c_3^2 . The exemplary values of the congruent parameter are in these regions, $1 \leq z \leq 3.53$ and $\pi/2 \leq \alpha \leq \pi/3.25$. Already within these ranges of congruent parameters, the bicubic formalism yields for particle limiting velocities that c_1 and c_2 are complex conjugate to each other, $c_1^* = c_2$, and that c_3 is imaginary. Furthermore, as long as z and α are kept fixed, the imaginary portions of c_1^2 and c_2^2 do not change the realities of numerically equal to each other dark matter energies $E(c_i)$, $i = 1; 2; 3$: In fact, real $E(c_{1,2})$ energies can be equally evaluated with $c_{1,2}^2$ or $\text{Re}c_{1,2}^2$ or even with $\text{Im}c_{1,2}^2$ so that in new notation, $E(c_{1,2}^2) = E(\text{Re}c_{1,2}^2) = E(\text{Im}c_{1,2}^2) = E(c_3^2)$ all with the same real values, with fixed values of $z(m)$ and related $\alpha(m)$. However, any individual of these energies' changes if just in it the simultaneous changes in $z(m)$ and $\alpha(m)$ are carried out, exhibiting the role of the particular limiting velocity square.

Keywords: Dark matter, Real energy, Complex limiting velocity.

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Josip Šoln*

JZS Phys-Tech, Vienna, Virginia 22182, USA.

(e-mail: jurasoln@yahoo.com)

**Corresponding Author*

I. INTRODUCTION

The so called dark matter particles do not seem to be successfully described so far by any physical models such as, for example, the Standard Model despite its large content. Here, based on the bicubic equation limiting particle velocity formalism as developed in [1]-[8] one is trying to understand dark matter particles with the formalism that is rather quite different from other descriptions. An explicit difference that follows from the

bicubic formalism is the fact that for three squares of limiting particle velocities, primary c_1^2 and secondary c_2^2 are not only complex but also complex conjugate to each other, while tertiary c_3^2 is real. Similarly, the three linear limiting particle velocities, primary c_1 and secondary c_2 are complex conjugate to each other while the tertiary c_3 is imaginary. The bicubic equation for particle limiting velocity, c , with global particle mass, m , velocity v and energy E , we take it to be in the same form as in [1] and [8]:

$$\left(\frac{c^2}{v^2}\right)^3 - \left(\frac{E}{mv^2}\right)^2 \left(\frac{c^2}{v^2}\right) + \left(\frac{E}{mv^2}\right)^2 = 0 \quad (1)$$

In our approach the cubic equation discriminant D , depending on the dimensionless real value congruent parameter, $z(m)$, is of the following global form [1]-[8] with restrictions as indicated,

$$D(m) = \left(\frac{27}{8}\right)^2 \frac{1}{z^4(m)} \left(1 - \frac{1}{z^2(m)}\right) \geq 0 \quad (2)$$

$$E = \frac{3\sqrt{3}mv^2}{2z(m)}, z^2(m) \geq 1 \quad (3)$$

The congruent parameter $z(m)$, appearing in (1), (2), (3), is nonlinearly related to the congruent angle $\alpha(m)$ (that follows) with which solutions $c_{1,2,3}^2$ from (1) can be expressed.

Section II contains the exact solutions for squares of limiting velocities c_1^2 , c_2^2 and c_3^2 , respectively of the primary, secondary and tertiary dark matter particles. Connecting the convenient squares of usual particle velocities, $v^2(c_i)$ with c_i^2 , $i = 1, 2, 3$, one can express the real particle energies in exact forms in terms of the newly introduced real congruent angle $\alpha(m)$, with two mutually consistent connections to the congruent parameter $z(m)$. Furthermore, here one establishes also three separate rather simple equivalent algebraic relations between $z(m)$ and $\alpha(m)$ which allow eliminating $z(m)$ in favor of $\alpha(m)$ making physical quantities to depend only on one parameter, the congruent angle $\alpha(m)$. These energies, despite involving some imaginary portions, are all real and of the same value, for fixed values of $z(m)$ and related $\alpha(m)$. The equal energies then change with simultaneous changes in $z(m)$ and $\alpha(m)$.

In Section III we describe developing of three linear particle limiting velocities, primary, c_1 , secondary, c_2 , and tertiary, c_3 , from the corresponding quadratic limiting velocity expressions by using the binomial equations ([9], p. 65).

In Section IV, with the elimination of the congruent parameter $z(m)$, one deals with samples of physical quantities with dark matter particle limiting velocities containing exclusively the congruent angle $\alpha(m)$.

Section V is devoted to summarizing the results and speculative assumptions. Evaluations of $c_{1,2}^2(m)$ and $c_3^2(m)$, as well as of c_1 , c_2 , and c_3 , respectively for congruent angle range from $\alpha = \pi/2.5$ to $\alpha = \pi/2$ are carried out. These are then compared with other possible angles, as well with some experimental data and results from [10]. In this Section also the squares of limiting velocity solutions $c_{1,2}^2(m)$ are applied to the dark matter particle believed to be

a sterile neutrino [3], [10] plus suggesting other physics attributes for lighter dark matter particles.

II. EXACT LIMITING VELOCITY SOLUTIONS IN QUADRATIC FORMS WITH CORRESPONDING PARTICLE ENERGIES

With the help of (2) and (3) the solutions of (1), are in somewhat simplified forms from [7], [8],

$$c_{1,2}^2(m) = \frac{2[1 \pm i\sqrt{3} \cos(\alpha(m))]\nu^2}{2z(m)\sin(\alpha(m))} = \text{Re } c_{1,2}^2(m) + i \text{Im } c_{1,2}^2(m) \quad (4)$$

$$\text{Re } c_{1,2}^2(m) = \frac{3\nu^2}{2z(m)\sin(\alpha(m))}, \quad (5)$$

$$\text{Im } c_{1,2}^2(m) = \pm \frac{3\sqrt{3}\nu^2}{2z(m)} \cot(\alpha(m))$$

$$c_3^2(m) = -\frac{3\nu^2}{z(m)\sin(\alpha(m))} \quad (6)$$

In the physics evaluations with limiting velocities c_1 , c_2 , and c_3 , associated respectively with primary, secondary and tertiary dark matter particles, a very useful parameter is the congruent angle $\alpha(m)$ nonlinearly related to the congruent parameter $z(m)$ [7], [8] with $z^2(m) \geq 1$, $0 < \alpha(m) \leq \pi/2$,

$$\alpha(m) = 2 \tan^{-1} \left(\tan \left(\frac{1}{2} \sin^{-1} \left(\frac{1}{z(m)} \right) \right) \right)^{\frac{1}{3}} \quad (7)$$

$$\frac{1}{z(m)} = \sin \left[2 \tan^{-1} \left(\tan \left(\frac{\alpha(m)}{2} \right) \right)^3 \right] \quad (8)$$

Both of them $z(m)$ and $\alpha(m)$, as we shall see shortly, have evolutionary roles on physical quantities. As long as the quantities such as E , m and ν together with $z(m)$ and $\alpha(m)$ are real, from (4), (5) and (6), one notices that $c_1^2(m)$ and $c_2^2(m)$ are complex conjugate to each other and $c_3^2(m)$ is negative and real, $c_{1,2}^{2*}(m) = c_{2,1}^2(m)$, $c_3^{2*}(m) = c_3^2(m)$. As we shall see later, linearization of limiting velocities will preserve these complex structures for $c_{1,2}^{*}(m) = c_{2,1}(m)$, but not for c_3 , $c_3^{*}(m) = -c_3$. When the congruent angle $\alpha(m) = \pi/2$, all $c_i^2(m)$, $i = 1, 2, 3$ are real. Also, with realities of $z(m)$ and $\alpha(m)$, from solutions (4), (5) and (6), one notices the zero sum rule for squares of limiting velocities [4], $c_1^2(m) + c_2^2(m) + c_3^2(m) = 0$, as well as $c_1^2(m)c_2^2(m) + c_1^2(m)c_3^2(m) + c_2^2(m)c_3^2(m) = -(E/m)^2$ which can be easily deduced by using the Cardano's formula ([9], p.65) and which indicates the correctness of c^2 solutions. From the combination of these two limiting

velocity squares one again deduces the dark matter particle energy E from (3).

Next, we wish to show that the congruent angle α and parameter z , in addition to (7) and (8), satisfy three more relations which will further reaffirm that $z(m)$ and $\alpha(m)$ are real. To do so, from solutions (4), (5) and (6) with the help of (3) one writes

$$\left(\frac{E}{mv^2}\right)^2 = \frac{27}{4z^2(m)}, z(m) \geq 1 \quad (9)$$

which is allowing to write three bicubic equations for the congruent parameter $z(m)$ itself.

When substituting (8) into (1) with solution notations from (4)-(6) one ends up with multiple expressions, vanishing separately for real and imaginary parts,

$$\text{Re, Im} \left[\left(\frac{c_{1,2,3}^2}{v^2}\right)^3 - \frac{27}{4z^2(m)} \left(\frac{c_{1,2,3}^2}{v^2}\right) + \frac{27}{4z^2(m)} \right] = 0 \quad (10)$$

Since the solutions for $c_{1,2,3}^2(m)$ are known from (4)-(6), relations (10) are two equations for $z(m)$ with $\alpha(m)$. Utilizing solutions (4)-(6) with some work, we start first with real part containing $c_{1,2}^2$:

$$\begin{aligned} \text{Re} \left[\left(\frac{c_{1,2}^2}{v^2}\right)^3 - \frac{27}{4z^2(m)} \left(\frac{c_{1,2}^2}{v^2}\right) + \frac{27}{4z^2(m)} \right] &= 0 \\ 1 - 9\cos(\alpha(m)) - 3\sin^2(\alpha(m)) + 2z(m)\sin^3(\alpha(m)) &= 0 \quad (11) \\ z(m) &= \frac{1 + 3\cos^2(\alpha(m))}{\sin^3(\alpha(m))} \end{aligned}$$

The resulting $z(m)$ is quite different in form from (8) but numerically the same, as we will be able to verify shortly.

Next, we address similarly the imaginary part with $c_{1,2}^2$ in (11), which with the help of (4), (5) yields simple result that $\alpha(m)$ is real.

$$\begin{aligned} \text{Im} \left[\left(\frac{c_{1,2}^2}{v^2}\right)^3 - \frac{27}{4z^2(m)} \left(\frac{c_{1,2}^2}{v^2}\right) + \frac{27}{4z^2(m)} \right] &= 0 \\ \mp \cos^2(\alpha(m)) \pm 1 \mp \sin^2(\alpha(m)) &= 0 \end{aligned} \quad (12)$$

Finally, we come to c_3^2 in (10) which with solution (6) yields the same result as in (11).

$$\begin{aligned} \left(\frac{c_3^2}{v^2}\right)^3 - \frac{27}{4z^2(m)} \left(\frac{c_3^2}{v^2}\right) + \frac{27}{4z^2(m)} &= 0 \\ -\frac{1}{\sin^3(\alpha(m))} + \frac{3}{4\sin(\alpha(m))} + \frac{z(m)}{4} &= 0 \quad (13) \\ z(m) &= \frac{4 - 3\sin^2(\alpha(m))}{\sin^3(\alpha(m))} \end{aligned}$$

The advantage of having (11) and (13) expressions for $z(m)$ is that now congruent parameter $z(m)$ can be eliminated in favor of just congruent angle $\alpha(m)$ in most of the expressions, if so desired. This can be summarized by also expressing the discriminant and the energy in just the terms of $\alpha(m)$:

$$z(m) = \frac{1 + 3\cos^2(\alpha(m))}{\sin^3(\alpha(m))} \geq 1, 0 < \alpha(m) \leq \frac{\pi}{2} \quad (14)$$

$$D(m) = \left(\frac{27}{8}\right)^2 \frac{\sin^{12}(\alpha(m))}{(1 + 3\cos^2(\alpha(m)))^4} \left(1 - \frac{\sin^6(\alpha(m))}{(1 + 3\cos^2(\alpha(m)))^2}\right)$$

$$E(m) = \frac{3\sqrt{3}\sin^3(\alpha(m))}{1 + 3\cos^2(\alpha(m))} \frac{mv^2}{2}, 0 < \alpha(m) \leq \frac{\pi}{2} \quad (15)$$

In Section IV, the primary, secondary and tertiary quadratic and, yet to discuss, linear limiting velocities for particles will be presented with just the congruent angle $\alpha(m)$. In general, the inter-related congruent parameters $z(m)$ and $\alpha(m)$ can play interchangeable roles as evolution parameters. In Table 1 calculated values are consistent with the relations (7)-(15).

TABLE 1: CONGRUENT PARAMETERS (m) AND $z(m)$ VALUES TO THIRD DECIMAL POINT APPROXIMATION

$\alpha(m)$	$\frac{\pi}{2}$	$\frac{\pi}{2.25}$	$\frac{\pi}{2.3}$	$\frac{\pi}{2.5}$	$\frac{\pi}{2.75}$	$\frac{\pi}{3}$	$\frac{\pi}{3.25}$
$\frac{1}{z(m)}$	1	0.876	0.835	0.669	0.496	0.371	0.283
$z(m)$	1	1.142	1.198	1.495	2.016	2.694	3.531

Later on, developing the three linear particle limiting velocities, primary, c_1 , secondary, c_2 , and tertiary c_3 , one requires three rather simple equivalent algebraic relations between $z(m)$ and $\alpha(m)$ that are the same as in (7)-(15) consistent with the values from Table 1 between $z(m)$ and $\alpha(m)$.

As shown in [8], it is illustrative to show the similarities and differences between our expression for energy from (3) and the relativistic energy expression (see [11]) by changing our congruent parameter $z(m)$ into $z(m; \text{relative})$ so that our energy from (3) becomes $E(\text{relative})$:

$$E(\text{relative}) = \frac{3\sqrt{3}mv^2}{2z(m, \text{relative})} = \frac{mv^2}{\beta^2(1-\beta^2)^{\frac{1}{2}}} \quad (16)$$

$$z(m, \text{relative}) = \frac{3\sqrt{3}}{2}\beta^2(1-\beta^2)^{\frac{1}{2}}, \quad \beta = \frac{v}{c} \quad (17)$$

$$z(m, \text{relative}) \lesssim 1, \beta \lesssim 1; \quad z(m, \text{relative}) \approx 1, \quad \beta \approx 0.8-0.84, \quad \alpha(m) \approx \frac{\pi}{2} \quad (18)$$

It is seen that in general $z(m)$ together with $\alpha(m)$ from (4)-(8), (13)-(15), may apply to subluminal, luminal or even superluminal particles because there are no prior restrictions on values of limiting velocities, which may be from subluminal to superluminal values. On the other hand with $z(m; \text{relative})$ as shown in (17) and (16) it would apply essentially to only subluminal particles with energy as in (16). The main reason for $z(m)$ to be expressed in terms of the congruent angle $\alpha(m)$ is to have more flexibility than when expressed in terms of β as in (17). This flexibility is needed if some of dark matter is of the superluminal type. From Table 1, it is evident how congruent parameter $z(m)$ differs from $z(m; \text{relative})$, particularly since one may have $z(m) > 1$ with limiting velocity still being the velocity of light c .

To continue in new different directions, we show how the squared dark matter particle velocity, v^2 , is directly related to squares of primary, secondary $\text{Re } c_{1,2}^2(\alpha(m))$, primary, secondary $\text{Im } c_{1,2}^2(\alpha(m))$, the whole primary, secondary $c_{1,2}^2(\alpha(m))$ and to tertiary $c_3^2(\alpha(m))$, respectively for each of the primary, secondary and tertiary dark matter particles. Taking into account that v^2 and $z(m)$ are real quantities, specifically from (4), (5) and (6) we derive the relations between squares of particle usual and limiting velocities, necessary in derivation of dark matter particle energies. To this end, we start by inverting limiting velocity solutions (4), (5) and (6),

$$c_{1,2}^2 : \frac{v^2}{z(m)} = \frac{2\sin(\alpha(m))c_{1,2}^2[1 \mp i\sqrt{3}\cos(\alpha(m))]}{3[1+3\cos^2(\alpha(m))]} \\ = \frac{2\sin(\alpha(m))}{3[1+3\cos^2(\alpha(m))]} [\text{Re } c_{1,2}^2 \pm \sqrt{3}\cos(\alpha(m))\text{Im } c_{1,2}^2] \\ + i(\text{Im } c_{1,2}^2 \mp \sqrt{3}\cos(\alpha(m))\text{Re } c_{1,2}^2) \quad (19)$$

$$\text{Reality} : i(\text{Im } c_{1,2}^2 \mp \sqrt{3}\cos(\alpha(m))\text{Re } c_{1,2}^2) = 0 \quad (20)$$

$$c_{1,2}^2 : \frac{v^2}{z(m)} = \frac{2\sin(\alpha(m))}{3[1+3\cos^2(\alpha(m))]} [\text{Re } c_{1,2}^2 \pm \sqrt{3}\cos(\alpha(m))\text{Im } c_{1,2}^2] \quad (21)$$

$$\text{Re } c_{1,2}^2 : \frac{v^2}{z(m)} = \frac{2}{3}\sin(\alpha(m))\text{Re } c_{1,2}^2 \quad (22)$$

$$\text{Im } c_{1,2}^2 : \frac{v^2}{z(m)} = \pm \frac{2}{3\sqrt{3}}\tan(\alpha(m))\text{Im } c_{1,2}^2 \quad (23)$$

$$c_3^2 : \frac{v^2}{z(m)} = -\frac{\sin(\alpha(m))}{3}c_3^2 \quad (24)$$

Let us point out that the reality condition (20) is already contained in (4) with (5). Here, with (20), together with real and imaginary parts from (4), (5) one transforms (19) into (21), (22) and (23) while (24) is due to the inversion of (6). With (21), (22), (23) and (24) we have complete quadratic limiting velocity presentations for evaluating respective primary, secondary and tertiary dark matter particle energies. To this effect, it is worthwhile to see how the values of the congruent angle $\alpha(m)$ from Table 1, according to the quadratic limiting velocity solutions (4), (5) and (6), may affect such calculations,

$$z(m) = 1, \quad \alpha(m) = \frac{\pi}{2} \\ : c_{1,2}^2 = \text{Re } c_{1,2}^2 = \frac{3}{2}v^2, \quad \text{Im } c_{1,2}^2 = 0, \quad c_3^2 = -3v^2 \quad (25)$$

$$z(m) = 1.495, \quad \alpha(m) = \frac{\pi}{2.5} \\ : c_{1,2}^2 = 1.055v^2 \pm i0.56v^2, \quad c_3^2 = -2.1v^2 \quad (26)$$

The dark matter particle energies labeled by specific limiting velocities, according to (3) together with relations (19)-(26) will follow. We start with the exemplary general expression:

$$E(c_{1,2,3}^2(m)) = \frac{3\sqrt{3}m}{2} \cdot \frac{v^2}{z(m)} (c_{1,2,3}^2(m)) \quad (27)$$

Next, in the fashion of (27), we deduce from (19) the following dark matter particle energies for the primary (c_1^2) and secondary (c_2^2) dark matter particles:

$$E(c_{1,2}^2(m)) = \frac{3\sqrt{3}m}{2} \cdot \frac{v^2}{z(m)} (c_{1,2}^2(m)) \\ = \sqrt{3}m\sin(\alpha(m))c_{1,2}^2(m) \frac{[1 \mp i\sqrt{3}\cos(\alpha(m))]}{[1+3\cos^2(\alpha(m))]} \\ = \frac{\sqrt{3}m3v^2[1 \pm i\sqrt{3}\cos(\alpha(m))][1 \mp i\sqrt{3}\cos(\alpha(m))]}{2z(m)[1+3\cos^2(\alpha(m))]} \\ \left(= \frac{3\sqrt{3}mv^2}{2z(m)} \right) \quad (28)$$

In this evaluation, the reality condition is automatically taken into account without specifying it in the course of evaluation with the solutions of limiting velocities (19)-(23). This is so, since from the limiting velocity solutions (4), (5) for the primary (c_1) and secondary (c_2) particles, one has implicitly that $\text{Im } c_{1,2}^2 = \pm \sqrt{3} \cos(\alpha(m)) \text{Re } c_{1,2}^2$. Similarly, we can evaluate the energy of the tertiary (c_3) particle from (24),

$$E(c_3^2(m)) = \frac{3\sqrt{3}m}{2} \cdot \frac{v^2}{z(m)} (c_3^2(m)) = -\frac{\sqrt{3}m \sin(\alpha(m)) c_3^2}{2} \left(= \frac{3\sqrt{3}mv^2}{2z(m)} \right) \quad (29)$$

All three of these, numerically equal, energies are real despite the fact that (primary, secondary) dark matter particle's squares of limiting velocities, $c_{1,2}^2$, are complex. Just knowing $\text{Re } c_{1,2}^2$ or $\text{Im } c_{1,2}^2$ one can still find out the corresponding energies of which each, as we shall see, equals the energy appearing in (28). Specifically, for $\text{Re } c_{1,2}^2$ and $\text{Im } c_{1,2}^2$ from respectively (22) and (23) we write:

$$E(\text{Re } c_{1,2}^2(m)) = \frac{3\sqrt{3}m}{2} \frac{v^2}{z(m)} \text{Re } c_{1,2}^2 = \sqrt{3}m \sin(\alpha(m)) \text{Re } c_{1,2}^2 \left(= \frac{3\sqrt{3}mv^2}{2z(m)} \right) \quad (30)$$

$$E(\text{Im } c_{1,2}^2(m)) = \frac{3\sqrt{3}m}{2} \frac{v^2}{z(m)} (\text{Im } c_{1,2}^2) = \pm m \tan \alpha(m) \text{Im } c_{1,2}^2 \left(= \frac{3\sqrt{3}mv^2}{2z(m)} \right) \quad (31)$$

The common value of energy $3\sqrt{3}mv^2/2z(m) = 3\sqrt{3}mv^2 \sin^3(\alpha(m))/2(1+3\cos^2(\alpha(m)))$ in (29-31) follows from solutions for (primary, secondary) $c_{1,2}^2(m)$, $\text{Re } c_{1,2}^2(m)$ and $\text{Im } c_{1,2}^2(m)$ in (4), (5) or (30), (31) and for (tertiary) $c_3^2(m)$ (6) limiting velocity squares. To change the energy, one changes the value of $z(m)$ or equivalently of $\alpha(m)$. The easiest way to exhibit that is to take specific congruent parameters with unspecified dark matter particle mass and velocity and inserting them into (28) or (30), (31):

$$\alpha(m) = \frac{\pi}{2.5}, \quad z(m) = 1.495$$

$$:\text{Re } c_{1,2}^2 = 1.055v^2, \quad \text{Im } c_{1,2}^2 = \pm 0.565v^2, \quad c_3^2 = -2.1v^2$$

$$E(c_{1,2}^2(m)) \approx 1.738mv^2, \quad E(\text{Re } c_{1,2}^2(m)) \approx 1.738mv^2$$

$$E(\text{Im } c_{1,2}^2(m)) \approx 1.739mv^2, \quad E(\text{Re } c_3^2(m)) \approx 1.729mv^2 \quad (32)$$

One may see the energy as a sacred quantity, as all these energy expressions, with fixed $z(m)$ and $\alpha(m)$, give the same value even from $\text{Im } c_1^2(m)$ (energy changes only if

$z(m)$ or equivalently $\alpha(m)$ changes). This particularly so, as for any energy, the zero sum rule for squares of limiting velocities holds for both real and imaginary ones.

$$\text{Re } c_1^2(m) + \text{Re } c_2^2(m) + c_3^2 = 0 \quad (33)$$

$$\text{Im } c_1^2(m) + \text{Im } c_2^2(m) = 0$$

As long as the congruent parameters satisfy $z(m) \neq 1$ and $\alpha(m) \neq \pi/2$, dark matter particle energies and momenta appear not to be expressible in the Lorentzian like forms as in [7]. However, as also in [7], they are expressible in more general usual forms. All the numerically equal energy expressions, including $E(\text{Im } c_{1,2}^2(m))$, are real, indicating energy as fundamental quantity in physics. The particle momentum is more a quantity of convenience, as its real value is associated with $\text{Re } c_1^2(m)$ or c_3^2 but not also with $\text{Im } c_1^2(m)$.

The dark matter particle momenta are defined with their energies, preferably in such a way, as to be real in values. With that in mind, we write down the particle momenta for primary, secondary and tertiary dark matter particles,

$$\bar{p}(c_{1,2}^2) = \bar{p}(\text{Re } c_{1,2}^2) = \frac{E(c_{1,2}^2(m))\bar{v}}{\text{Re } c_{1,2}^2} = \sqrt{3}m\bar{v} \sin(\alpha(m))$$

$$\bar{p}(c_3^2) = \frac{E(c_3^2(m))\bar{v}}{(-c_3^2(m))} = \frac{\sqrt{3}}{2}m\bar{v} \sin(\alpha(m)), \quad 0 < \alpha(m) \leq \frac{\pi}{2} \quad (34)$$

For primary, secondary dark matter particles with $c_{1,2}$ limiting velocities, in the definitions we use (primary, secondary) $\text{Re } c_{1,2}^2(m)$ with $E \text{Re } c_{1,2}^2(m)$ (numerically equal to $E c_{1,2}^2(m)$) to define the equal value momenta of which each in form is very similar to tertiary particle momentum but double in value. What we have here is basically the same energy particle but with variety of physical attributes.

III. LINEAR FORMS OF COMPLEX LIMITING VELOCITIES IN DESCRIPTION OF DARK MATTER PARTICLES

The linearization of particle limiting velocities from (4), (5) will be done in analog of linearizing of these quadratic bionomical equations yielding, with the definition of r , by trial the linearized $d(+, -)$ as indicated ([9], p.65) :

$$d^2(+, -) = a + i(\pm b), \quad r = \sqrt{a^2 + (\pm b)^2} = \sqrt{a^2 + b^2}$$

$$d(+) = \pm \left[\sqrt{\frac{r+a}{2}} + i\sqrt{\frac{r-a}{2}} \right]$$

$$d(-) = \pm \left[\sqrt{\frac{r+a}{2}} - i\sqrt{\frac{r-a}{2}} \right] \quad (35)$$

whose verification backward is easy to carry out. The

linearization of $c_1^2(m)$ and $c_2^2(m)$ from (4) and (5) will be done respectively with substitutions

$$d(+) = c_1(m), d(-) = c_2(m) \quad (36)$$

$$a = \frac{3v^2}{2z(m)\sin(\alpha(m))}, b = \frac{3\sqrt{3}ctm(\alpha(m))v^2}{2z(m)} \quad (37)$$

$$r^2 = \left(\frac{3v^2}{2z(m)\sin(\alpha(m))} \right)^2 (1 + 3\cos^2(\alpha(m))) \quad (38)$$

With proper substitutions from (36)-(38) into (35) one arrives at linear expressions of primary, secondary and also, directly from (6), tertiary limiting velocities, c_1 , c_2 , and c_3 .

$$c_{1,2}(m) = (\pm v) \left(\frac{3}{4z(m)\sin(\alpha(m))} \right)^{1/2} \times \left[\left(1 + \sqrt{1 + 3\cos^2(\alpha(m))} \right)^{1/2} + i, -i \left(\sqrt{1 + 3\cos^2(\alpha(m))} - 1 \right)^{1/2} \right] \quad (39)$$

$$c_3(m) = i(\pm v) \sqrt{\frac{3}{z(m)\sin(\alpha(m))}}, 0 < \alpha(m) \leq \frac{\pi}{2} \quad (40)$$

These, when squared become exactly (4)-(6). One notices preservations of complexities for $c_1(m)$ and $c_2(m)$, $c_{1,2}^*(m) = c_{2,1}(m)$, but not the reality for $c_3(m)$, $c_3^*(m) = -c_3(m)$. The simple sample values of congruent quantities $\alpha(m)$ and $z(m)$ from (25) and (26), $\alpha(m) = \pi/2, \pi/2.5$ with respective $z(m) = 1, 1.495$, from numerical evaluations of quadratic limiting velocities $c_{1,2}^2(m)$, are also used here when evaluating linear limiting velocities $c_{1,2}(m)$ and $c_3(m)$ from (39) and (40),

$$z(m) = 1, \alpha(m) = \frac{\pi}{2} \quad (41)$$

$$: c_{1,2} = \text{Re } c_{1,2} = (\pm v) \sqrt{\frac{3}{2}}, \text{Im } c_{1,2} = 0, c_3 = i(\pm v) \sqrt{3}$$

$$z(m) = 1.495, \alpha(m) = \frac{\pi}{2.5} \quad (42)$$

$$: c_{1,2} = (\pm v)(1.061 + i, -i0.266), c_3 = i(\pm v)1.45$$

One notices the numerical consistencies between respective relations of (41), (42) with (25), (26), by squaring (41) and (42) and taking into account that in general,

$$\text{Re } c_{1,2}^2 = (\text{Re } c_{1,2})^2 - (\text{Im } c_{1,2})^2 \quad (43)$$

$$\text{Im } c_{1,2}^2 = 2 \text{Re } c_{1,2} \text{Im } c_{1,2}, c_3^2 = (c_3)^2 \quad (44)$$

The relations (43) and (44), although simple, in fact are important to take into account, when operating with linear form limiting velocities $c_{1,2}$, but evaluating physical quantities depending on either $c_{1,2}^2$ or $\text{Re } c_{1,2}^2$ and even on $\text{Im } c_{1,2}^2$. Relation (43) is also good example of how even imaginary limiting velocity expressions can contribute to the real ones and to the real results.

IV. JUST CONGRUENT ANGLE $\alpha(m)$ DESCRIPTION OF DARK MATTER PARTICLE LIMITING VELOCITIES

Working in Section II on details of the exact solutions of particle limiting velocities (4)-(6), the necessity of the congruent angle $\alpha(m)$, related to the congruent parameter $z(m)$, got firmly established. Their rather strong non-linear relationships are already given in relations (7) and (8). However, studying real and imaginary parts of particle limiting velocity expressions, a relatively simpler nonlinear relationship between $z(m)$ and $\alpha(m)$ got established, as shown in (14). The big advantage of this expression for the dark matter particle energy is in the fact that it does not involve directly $\text{Re } c_{1,2}^2(m)$ and $\text{Im } c_{1,2}^2(m)$, which was already carried out in relations (27) to (32). Some practical aspects will be mentioned briefly also in the Section V.

Here, however using the new expression (14) for $z(m)$ in terms of $\alpha(m)$ we turn to the dark matter particle limiting velocities, in both, quadratic and linear forms as expressions in terms of $\alpha(m)$. First, for limiting velocities in quadratic forms,

$$(4) : c_{1,2}^2(m) = \frac{3[1 \pm i\sqrt{3}\cos(\alpha(m))]v^2\sin^2(\alpha(m))}{2(1 + 3\cos^2(\alpha(m)))} \quad (45)$$

$$= \text{Re } c_{1,2}^2(m) + i \text{Im } c_{1,2}^2(m), 0 < \alpha(m) \leq \frac{\pi}{2}$$

$$(5) : \text{Re } c_{1,2}^2(m) = \frac{3v^2\sin^2(\alpha(m))}{2(1 + 3\cos^2(\alpha(m)))}, 0 < \alpha(m) \leq \frac{\pi}{2}$$

$$\text{Im } c_{1,2}^2(m) = \pm \frac{3\sqrt{3}v^2\sin^2(\alpha(m))\cos(\alpha(m))}{2(1 + 3\cos^2(\alpha(m)))}, 0 < \alpha(m) \leq \frac{\pi}{2} \quad (46)$$

$$c_3(m) = -\frac{3v^2\sin^2(\alpha(m))}{(1 + 3\cos^2(\alpha(m)))}, 0 < \alpha(m) \leq \frac{\pi}{2} \quad (47)$$

Second, for limiting velocities in linear forms expressed in terms of $\alpha(m)$,

$$(39), (40) : c_{1,2}(m) = (\pm v) \frac{\sqrt{3} \sin(\alpha(m))}{2} \times \left[\left(\frac{1 + \sqrt{1 + 3 \cos^2(\alpha(m))}}{1 + 3 \cos^2(\alpha(m))} \right)^{1/2} \pm i \left(\frac{\sqrt{1 + 3 \cos^2(\alpha(m))} - 1}{1 + 3 \cos^2(\alpha(m))} \right)^{1/2} \right]$$

$$c_3(m) = i(\pm v) \sqrt{3} \frac{\sin(\alpha(m))}{\sqrt{1 + 3 \cos^2(\alpha(m))}}, 0 < \alpha(m) \leq \frac{\pi}{2} \quad (48)$$

The energy expression (15) from quadratic forms limiting velocities, shows that the congruent angle $\alpha(m)$ is an evolutionary parameter whose changing value from $\pi/2$ to 0 changes the energy E from $3\sqrt{3}mv^2/2$ to 0.

The new expression (14) for the congruent parameter $z(m)$ when applied respectively to (7) and (8) yield two amusing (but correct) self-identities, for $0 < \alpha(m) \leq \frac{\pi}{2}$:

$$(7) : \alpha(m) = 2 \tan^{-1} \left(\tan \left(\frac{1}{2} \sin^{-1} \left(\frac{\sin^3(\alpha(m))}{1 + 3 \cos^2(\alpha(m))} \right) \right) \right)^{\frac{1}{3}} \quad (49)$$

$$(8) : \frac{\sin^3(\alpha(m))}{(1 + 3 \cos^2(\alpha(m)))} = \sin \left[2 \tan^{-1} \left(\tan \left(\frac{\alpha(m)}{2} \right) \right)^3 \right] \quad (50)$$

which are the results of more than one representation of the congruent parameter $z(m)$ in terms of the congruent angle $\alpha(m)$.

V. DISCUSSION WITH EXAMPLE APPLICATIONS AND CONCLUSION

One notices interesting things happening to dark matter particles once the congruent parameter satisfies $z(m) \neq 1$ and congruent angle $\alpha(m) \neq \pi/2$. As one sees, the nonlinearly connected dimensionless congruent parameter $z(m)$ and dimensionless congruent angle $\alpha(m)$ are essential in evaluating not only all forms of particle energies but also of linear particle momenta. In fact, as we can see from (16)-(18), they have similar roles in evaluating particle energy E as does the relative velocity from Special Relativity. For one thing, the Lorentzian like form is not favored by either the dark matter particle energy or the momentum as seen respectively in each of (28)-(33) and (34). The most amazing things are that different forms of dark matter particle complex limiting velocity-squares (primary, secondary) $c_{1,2}^2(m)$, (primary, secondary) $\text{Re } c_{1,2}^2(m)$ and (primary, secondary) $\text{Im } c_{1,2}^2(m)$ (4), (5) separately yield the same value dark matter energy $E(c_{1,2}^2(m))$, while the real (tertiary) $c_3^2(m)$ (6) yields $E(c_3^2(m))$ (numerically equal to $E(c_{1,2}^2(m))$). Similarly, one

has the expressions for dark matter particle momenta, $\vec{p}(c_{1,2}^2) = \vec{p}(\text{Re } c_{1,2}^2)$ from $E(c_{1,2}^2(m))$ together with (primary, secondary) $\text{Re } c_{1,2}^2(m)$, while tertiary dark matter particle momentum $\vec{p}(c_3^2)$ follows in a usual way from limiting velocity-square $c_3^2 m$ and the energy $E(c_3^2(m))$.

Recently Kenny C. Y. Ng et al. [10], from studying dark matter from NuS-TAR M31 observations have put forward well-motivated dark sterile neutrino dark matter candidate, denoted as χ , which radioactively can decay into monoenergetic photon γ , plus active neutrino ν , $\chi \rightarrow \gamma + \nu$. Here, we wish to move on from the congruent parameters values of $z(m) = 1$ and $\alpha(m) = \pi/2$, as in [8], to different values with $z(m) \neq 1$ and $\alpha(m) \neq \pi/2$. In doing so, also here as in [8] with [10], we shall accept that the dark sterile neutrino mass satisfies $m_\chi \geq 12 \text{ keV} / c^2$, with c the velocity of light. Continuing with the example of the dark sterile neutrino candidate, we shall demonstrate the evolutionary qualities of interrelated congruent angle $\alpha(m_\chi)$ and congruent parameter $z(m_\chi)$ with decreasing values of $\alpha(m_\chi) = \pi/2.5, \pi/2.3$ and $\pi/2$, and corresponding increasing values of $z(m_\chi)$ according to (14). The purpose here is also to demonstrate the usefulness of relation (43), (44), allowing to evaluate quantities depending on $c_{1,2}^2$ and c_3^2 , but starting from linear limiting velocities $c_{1,2}$ and c_3 .

Hence, we can start with linear limiting velocities $c_{1,2}$ and c_3 , from (39) and (40), depending on both $z(m_\chi)$ and $\alpha(m_\chi)$, or from (45), depending only on $\alpha(m_\chi)$ yielding the same values for linear velocities. Specifically, from (48) we derive linear limiting dark matter velocities, $\text{Re } c_{1,2}$, $\text{Im } c_{1,2}$ and c_3 in terms of the dark matter velocity v , each matched with the corresponding congruent angle $\alpha(m_\chi)$.

$$m_\chi = 12 \text{ keV} / c^2$$

$$\alpha(m_\chi) = \pi/2.5 : \text{Re } c_{1,2} = 1.061v, \text{Im } c_{1,2} = \pm 0.2166v, c_3 = i1.452v$$

$$\alpha(m_\chi) = \pi/2.3 : \text{Re } c_{1,2} = 1.148v, \text{Im } c_{1,2} = \pm 0.194v, c_3 = i1.599v$$

$$\alpha(m_\chi) = \pi/2 : \text{Re } c_{1,2} = 1.225v, \text{Im } c_{1,2} = 0v, c_3 = i1.732v$$

As relations (27) to (34) indicate also quadratic form limiting velocities can be used directly to evaluate physical quantities. These are easy to obtain from the linear ones from (43), (44) yielding now from (45)-(47) the necessary quadratic forms:

$$m_\chi = 12keV / c^2$$

$$\alpha(m_\chi) = \pi / 2.5 : \text{Re } c_{1,2} = 1.055v^2, \text{ Im } c_{1,2} = \pm 0.565v^2, c_3 = -2.110v^2$$

$$\alpha(m_\chi) = \pi / 2.3 : \text{Re } c_{1,2} = 1.279v^2, \text{ Im } c_{1,2} = \pm 0.451v^2, c_3 = -2.551v^2$$

$$\alpha(m_\chi) = \pi / 2 : \text{Re } c_{1,2} = 1.5v^2, \text{ Im } c_{1,2} = \pm 0v^2, c_3 = -3v^2$$

Next, the evaluation of dark energies is in order. The dark matter particle energy for any limiting velocity form, but of desired congruent parameter $z(m_\chi)$ with corresponding congruent angle $\alpha(m_\chi)$, one can evaluate according to (27) to (31), in which both $z(m_\chi)$ and $\alpha(m_\chi)$, are involved. Equivalently the same energy is evaluated directly from (15) involving just $\alpha(m_\chi)$, which, being the evolutionary parameter, determines the energy in parallels with limiting velocities.

$$m_\chi = 12keV / c^2$$

$$\alpha(m_\chi) = \pi / 2.5 : E(m_\chi) = 20.487keV \frac{v^2}{c^2}$$

$$\alpha(m_\chi) = \pi / 2.3 : E(m_\chi) = 26.029keV \frac{v^2}{c^2}$$

$$\alpha(m_\chi) = \pi / 2 : E(m_\chi) = 31.177keV \frac{v^2}{c^2}$$

In comparison, we also calculate an energy from expressions involving dark matter (sterile neutrino) particle limiting velocities (28)-(31) with fixed congruent angle of $\alpha(m_\chi) = \pi / 2.5$ which with (13) yields $z(m_\chi) = 1.2955$. Then $m_\chi = 12keV / c^2$ and according to (28)-(31) yields the same energy for variety of squared limiting velocities:

$$E(c_{1,2}^2) = E(\text{Re } c_{1,2}^2) = E(\text{Im } c_{1,2}^2) = E(c_3^2) = \frac{3\sqrt{3}m_\chi}{2z(m_\chi)} = 20.847keV \frac{v^2}{c^2}$$

Similar analyses one can perform for dark sterile neutrino momenta but restricting to just the real ones (34).

In conclusion, the fact that through the bicubic equation limiting particle velocity formalism the real particle energy can be equally well evaluated from the complex, real or imaginary particle limiting velocity-squared expression, makes the particle energy exceptional and important, almost, a unique natural quantity, particularly as relations (32) shows explicitly in succession numerically equal real values $E(c_{1,2}^2) = E(\text{Re } c_{1,2}^2) = E(\text{Im } c_{1,2}^2) = E(c_3^2)$ for each velocity-squared, complex c_1^2 , complex c_2^2 and real c_3^2 . What this shows that one should not automatically discard complex or imaginary quantities in physics as their contents may support real physical quantities, such as energy. The linear forms of particle limiting velocities are not less important as their comparisons to the velocity of light may show unique

the velocity of light really is.

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