European Journal of Applied Physics
Vol 6 | Issue 5 | October 2024
ISSN 2684-4451

RESEARCH ARTICLE

Entanglement of Coupled Harmonic Oscillators without and with Tunneling Effect and Correction Factor

Ahlem Abidi*

ABSTRACT

Seeing its simplicity of processing, the system of two coupled harmonic oscillators have witnessed a quick growth of research conducted to quantum information. This approach is explicitly investigated in this paper, in particularly entanglement concept is proved to be intimately related to the tunneling effect. We examine analytically entanglement dynamics by introducing the Lewis and Riesenfeld invariant operator in the Heisenberg picture approach to compute the density matrix on the based of an exact treatment. We use Wigner function of the mixed state as an essential tool to move to linear entanglement entropies and we compute the corresponding correction factor. We follow numerically the evolution of entanglement dynamics without and under tunneling through the quantum potential barrier by introducing two particular models between simple and damped coupled harmonic oscillators. Entanglement dynamics is considered to be average without tunneling, it grows upon encountering the potential barrier and remains moderately constant inside barrier. This specificity is extended for both models but with higher values for damped coupled harmonic oscillators consequently the damping effect rapidly increases entanglement. Correction factor is also considered for both models, it show that; low temperature and high potential barriers make the system more disruptive. An increase of the coupling parameter of the system increase correction factor. Damping make the correction factor more important so damping disturbs more the system. Interference effect increase correction factor and it shows an interference between the values of the barrier penetration integral.

Keywords: Correction factor, coupled harmonic oscillators system, entanglement, Tunneling effect.

Submitted: April 23, 2024

Published: October 13, 2024

10.24018/ejphysics.2024.6.5.315

Tunis El Manar University, Faculty of Sciences of Tunis, Department of Physics, Laboratory of Nanostructured Materials, Quantum and Nonlinear Optics.

Corresponding Author:
e-mail: abidiiahlem0@gmail.com

1. Introduction

Quantum description of a system is intimately linked to the way it is perceived. Considering thus; the system of two coupled harmonic oscillators, this model is applied in biophysics [1], molecular physics and chemistry [2], [3], quantum physics [4]–[6]... etc. The dynamics of such system is determined by Schrödinger equation $i\frac{\partial\psi}{\partial t} = H\psi$ involving the Hamiltonian operator H. In various situations, the physical parameters in the expression of the Hamiltonian are time-dependent. The study of such Hamiltonians is very important on the modeling of real physical systems [7]–[10]. Specifying quantum information, their conception is currently a key focus of study such as in quantum cryptography [11], [12], quantum computing algorithms [13], and quantum coding [14]. Another description of quantum mechanics that does not have a classical analog connected to particles, is the tunneling effect. It is defined by the ability of a particle to cross an impenetrable classical barrier. This phenomenon is used in optical physics [15], [16], chemistry [17], [18], and biology [19]. Our interest in this paper is quantum entanglement of non-stationary mixed states and the impact of the tunneling effect on it. The discussion is focused counterpart the solutions of the classical equations of motion. In order to be more explicit, see references [20]–[22]. To take full advantage of tunneling effect on quantum information, we could briefly prepare Hamiltonian of interest to a diagonal form. We interposed the invariant operator constructed by the Lewis and Riesenfeld method to get a full dynamic description. We derive entanglement entropies and we compute the corresponding correction factor. We tracked the evolution of entanglement, without tunnel effect, at the meeting of the potential barrier, inside it; the evolution of the correction factor by following particular parameters of the system and the impact of the interference effect on these two concepts.

2. Heisenberg Picture Approach to Entangled System

Time-dependent hamiltonian consisting of two coupled harmonic oscillators is introduced as follows:

$$H(t)=\frac{1}{2M(t)}\left(p_1^2(t)+p_2^2(t)\right)+\frac{1}{2}M(t)\Omega_1(t)y_1^2(t)+\frac{1}{2}M(t)\Omega_2(t)y_2^2(t)-\frac{1}{2}\lambda(t)\left(y_1(t)-y_2(t)\right)^2, \quad (2.1)$$

Two harmonic oscillators with angular frequencies $\Omega_j(t)$. $y_j(t)$ are the time-dependent coordinates position and $p_j(t)=-i\hbar\frac{\partial}{\partial y_j}$ are the coordinates momentum of $y_j(t)$, checking the relation $\left[y_i(t),p_j(t)\right]=i\delta_{ij}$ with $(i,j=1,2)$. $\lambda(t)$ is the spatial coupling parameter. As usual, in terms of the new variables $\hat{q}_1=\cos\gamma y_1-\sin\gamma y_2$, $\hat{q}_2=\sin\gamma y_1+\cos\gamma y_2$, $P_1=\cos\gamma p_1-\sin\gamma p_2$, $P_2=\sin\gamma p_1+\cos\gamma p_2$ and we have set the rotation angle as $2\gamma=\tan^{-1}\left(\frac{\lambda(t)}{M(t)\left(\Omega_2(t)-\Omega_1(t)\right)}\right)$, we can reach the diagonal form

$$\begin{aligned}H(t)=&\frac{1}{2M(t)}\left(P_1^2(t)+P_2^2(t)\right)+\frac{1}{2}\left(\xi_1(t)\hat{q}_1^2(t)+\xi_2(t)\hat{q}_2^2(t)\right)\\&=H_1(t)+H_2(t),\end{aligned} \quad (2.2)$$

Expression (2.2) is the Hamiltonian of two decoupled harmonic oscillators. Notice that

$$\xi_1(t)=M(t)\Omega_1(t)-\lambda(t)-\frac{\lambda(t)}{M(t)}\tan\gamma, \textit{and } \xi_2(t)=M(t)\Omega_1(t)-\lambda(t)+\frac{\lambda(t)}{M(t)}\tan\gamma. \quad (2.3)$$

In accordance to (2.2), the related Lewis and Riesenfeld invariant by referring to [23] is provided as:

$$\begin{aligned}I(t)=&\frac{1}{2}\left(\bar{P}_1^2(t)+\bar{P}_2^2(t)\right)+\frac{1}{2}\left(\bar{\omega}_{I1}(t)\hat{Q}_1^2(t)+\bar{\omega}_{I2}(t)\hat{Q}_2^2(t)\right)\\&=I_1(t)+I_2(t),\end{aligned} \quad (2.4)$$

where

$$\bar{\omega}_{Ij}(t)=g_{+j}(t)g_{-j}(t)-g_{0j}^2(t). \quad (2.5)$$

Expression $g_{-j}(t)$ in (2.5) take the general forms

$$g_{-j}(t)=c_{1j}f_{1j}^2+c_{2j}f_{1j}f_{2j}+c_{3j}f_{2j}^2, \quad (2.6)$$

which are f_{kj} solutions of the classical functions of motion

$$\frac{d}{dt}\left(\dot{f}_{kj}(t)\right)+\frac{\dot{M}(t)}{M(t)}\dot{f}_{kj}(t)+\sqrt{\xi_k(t)}f_{kj}(t)=0,$$

where $(k=1.2)$ and g_{0j},g_{+j} are given by a direct differentiation of the following forms

$$g_{0j}(t)=-\frac{M(t)}{2}\frac{dg_{-j}(t)}{dt}, \quad (2.7)$$

and

$$g_{+j}(t)=M^2(t)\xi_j^2(t)\frac{dg_{-j}(t)}{dt}-M(t)\frac{dg_{0j}(t)}{dt}. \quad (2.8)$$

Variables of (2.4) are related to variables of (2.2) as

$$\bar{P}_j=\sqrt{g_{-j}(t)}\left(P_j+\frac{g_{0j}(t)}{g_{-j}(t)}\hat{q}_j\right), \textit{ and } \hat{Q}_j=\frac{1}{\sqrt{g_{-j}(t)}}\hat{q}_j.$$

Now comparing the invariant of expression (2.4) and the Hamiltonian of expression (2.2); use the procedures of [24] and the orthogonality relation

$$\int |\hat{Q}_1, \hat{Q}_2\rangle\langle\hat{Q}_1, \hat{Q}_2| d\hat{Q}_1 d\hat{Q}_2 = I,$$

consequently we obtain the density matrix:

$$\rho\left(\hat{Q}_{1f}, \hat{Q}_{2f}; \hat{Q}_{1i}, \hat{Q}_{2i}\right) = \int d\hat{Q}_1 d\hat{Q}_2 \exp\left[\frac{i}{2}\int_{t_0}^{t_f}\left(\dot{\hat{Q}}_1^2 + \dot{\hat{Q}}_2^2 - \left(\bar{\omega}_{I1}(t)\hat{Q}_1^2(t) + \bar{\omega}_{I2}(t)\hat{Q}_2^2(t)\right)\right)dt\right]. \quad (2.9)$$

We can rewrite it as

$$\rho\left(\hat{Q}_{1f}, \hat{Q}_{2f}; \hat{Q}_{1i}, \hat{Q}_{2i}\right) = \rho_1\left(\hat{Q}_{1f}, \hat{Q}_{1i}\right)\rho_2\left(\hat{Q}_{2f}, \hat{Q}_{2i}\right). \quad (2.10)$$

i and f denote respectively initial and later time.

It is then convenient to write the reduced density matrices of the first and the second harmonic oscillator as

$$\rho_1\left(\hat{Q}_{1f}, \hat{Q}_{1i}\right) = \left(\frac{1}{\det(2\pi i C_{1fi})}\right)^{\frac{1}{2}} \exp\left[\frac{i}{2}\left(\frac{B_{1fi}}{C_{1fi}}\hat{Q}_{1f}^2 + \frac{A_{1fi}}{C_{1fi}}\hat{Q}_{1i}^2 - 2\frac{1}{C_{1fi}^2}\hat{Q}_{1i}\hat{Q}_{1f}\right)\right], \quad (2.11)$$

and

$$\rho_2\left(\hat{Q}_{2f}, \hat{Q}_{2i}\right) = \left(\frac{1}{\det(2\pi i C_{2fi})}\right)^{\frac{1}{2}} \exp\left[\frac{i}{2}\left(\frac{B_{2fi}}{C_{2fi}}\hat{Q}_{2f}^2 + \frac{A_{2fi}}{C_{2fi}}\hat{Q}_{2i}^2 - 2\frac{1}{C_{2fi}^2}\hat{Q}_{2i}\hat{Q}_{2f}\right)\right]. \quad (2.12)$$

With

$$\hat{Q}_{jf} = \frac{1}{\sqrt{g_{-j}(t)}}\hat{q}_j(t_f), \; and \; (j = 1, 2) \quad (2.13)$$

By referring to expressions (2.2) and (2.4), we can set the initial condition as

$$I(t_0) = H(t_0). \quad (2.14)$$

Consequently, we write

$$g_{-j}(t) = \frac{1}{M(t_0)}, \quad g_{0j}(t) = 0 \text{ and } g_{+j}(t) = M(t_0)\Omega_j(t). \quad (2.15)$$

At initial time t_0, we define the variables $\hat{Q}_j$ by

$$\hat{Q}_{ji} = \frac{1}{\sqrt{M(t_0)}}\hat{q}_j(t_0). \quad (2.16)$$

Expression (2.10) can then be written of the original variables $\hat{q}_j$ as

$$\rho\Big(\hat{q}_2(t_f), \hat{q}_{2i}(t_0); \hat{q}_{1f}(t_f), \hat{q}_{1i}(t_0)\Big) = \left(\frac{1}{\det\left(2\pi i C_{1fi}\right)\det\left(2\pi i C_{2fi}\right)}\right)^{\frac{1}{2}}$$
$$\times \exp\left[\frac{i}{2}\Big\{-\alpha_{1f}\hat{q}_1^2(t_f) - \alpha_{1i}\hat{q}_1^2(t_0) - \alpha_{2f}\hat{q}_2^2(t_f) - \alpha_{2i}\hat{q}_2^2(t_0)\Big\}\right]$$
$$\times \exp\left[\frac{i}{2}\Big\{2\beta_{1f}\hat{q}_1(t_f)\hat{q}_2(t_f) + \beta_{1i}\hat{q}_1(t_0)\hat{q}_2(t_0) + 2\eta_1\hat{q}_1(t_f)\hat{q}_1(t_0) + 2\eta_2\hat{q}_2(t_f)\hat{q}_2(t_0) - 2\vartheta\hat{q}_1(t_0)\hat{q}_2(t_f) - 2\vartheta\hat{q}_1(t_f)\hat{q}_2(t_0)\Big\}\right], \tag{2.17}$$

then, we have

$$\alpha_{1f} = -\frac{1}{2}\left[\frac{1}{\sqrt{g_{-1}(t)}}\frac{B_{1fi}}{C_{1fi}}\cos^2(\gamma) + \frac{1}{\sqrt{g_{-2}(t)}}\frac{B_{2fi}}{C_{2fi}}\sin^2(\gamma)\right], \tag{2.18}$$

$$\alpha_{1i} = -\frac{1}{2M(t_0)}\left[\frac{A_{1fi}}{C_{1fi}}\cos^2(\gamma) + \frac{A_{2fi}}{C_{2fi}}\sin^2(\gamma)\right], \tag{2.19}$$

$$\alpha_{2f} = -\frac{1}{2}\left[\frac{1}{\sqrt{g_{-1}(t)}}\frac{B_{1fi}}{C_{1fi}}\sin^2(\gamma) + \frac{1}{\sqrt{g_{-2}(t)}}\frac{B_{2fi}}{C_{2fi}}\cos^2(\gamma)\right], \tag{2.20}$$

$$\alpha_{2i} = -\frac{1}{2M(t_0)}\left[\frac{A_{1fi}}{C_{1fi}}\sin^2(\gamma) + \frac{A_{2fi}}{C_{2fi}}\cos^2(\gamma)\right], \tag{2.21}$$

$$\beta_{1f} = -\left[\frac{1}{\sqrt{g_{-1}(t)}}\frac{B_{1fi}}{C_{1fi}} - \frac{1}{\sqrt{g_{-2}(t)}}\frac{B_{2fi}}{C_{2fi}}\right]\sin(\gamma)\cos(\gamma), \tag{2.22}$$

$$\beta_{2f} = -\frac{1}{\sqrt{M(t_0)}}\left[\frac{A_{1fi}}{C_{1fi}} - \frac{A_{2fi}}{C_{2fi}}\right]\sin(\gamma)\cos(\gamma), \tag{2.23}$$

$$\eta_1 = -\frac{1}{\sqrt{M(t_0)}}\left[\frac{1}{\sqrt{g_{-1}(t)}}\frac{1}{C_{1fi}^2}\cos^2(\gamma) + \frac{1}{\sqrt{g_{-2}(t)}}\frac{1}{C_{2fi}^2}\sin^2(\gamma)\right], \tag{2.24}$$

$$\eta_2 = -\frac{1}{\sqrt{M(t_0)}}\left[\frac{1}{\sqrt{g_{-1}(t)}}\frac{1}{C_{1fi}^2}\sin^2(\gamma) + \frac{1}{\sqrt{g_{-2}(t)}}\frac{1}{C_{2fi}^2}\cos^2(\gamma)\right], \tag{2.25}$$

$$\vartheta = -\frac{1}{\sqrt{M(t_0)}}\left[\frac{1}{\sqrt{g_{-1}(t)}}\frac{1}{C_{1fi}^2} + \frac{1}{\sqrt{g_{-2}(t)}}\frac{1}{C_{2fi}^2}\right]\sin(\gamma)\cos(\gamma), \tag{2.26}$$

and

$$A_{jfi} = \sqrt{M(t)}\cos\sqrt{\bar{\omega}_{Ij}(t)}T\frac{1}{\sqrt{M(t)}}, \tag{2.27}$$

$$B_{jfi} = \frac{1}{\sqrt{M(t)}}\cos\sqrt{\bar{\omega}_{Ij}(t)}T\sqrt{M(t)}, \tag{2.28}$$

$$C_{jfi} = \frac{1}{\sqrt{M(t)}}\frac{\sin\sqrt{\bar{\omega}_{Ij}(t)}T}{\sqrt{\bar{\omega}_{Ij}(t)}}\frac{1}{\sqrt{M(t)}}, \tag{2.29}$$

where $T = t_f - t_0$.

To move on to the entanglement process, we have selected a simple path connecting the density matrix to the Wigner function of mixedness by the relation [25]

$$W\left(\hat{q}_1,\hat{q}_2;P_1,P_2\right)=\frac{1}{2\pi}\int\rho\left(\hat{q}_1-\frac{\tau_1}{2},\hat{q}_2-\frac{\tau_2}{2};\hat{q}_1+\frac{\tau_1}{2},\hat{q}_2+\frac{\tau_2}{2}\right)e^{iP_1\tau_1+iP_2\tau_2}d\tau_1d\tau_2, \tag{2.30}$$

and the linear entanglement entropy with

$$S=1-(2\pi)^2\int W^2d\hat{q}dP. \tag{2.31}$$

We make the variables at an arbitrary time such that

$$\left(\hat{q}_1(t_0),\hat{q}_2(t_0)\right)=\left(\hat{q}_1(t_f),\hat{q}_2(t_f)\right)=\left(\hat{q}_1(t),\hat{q}_2(t)\right), \tag{2.32}$$

expression (2.30) yields

$$\begin{aligned}W\left(\hat{q}_1,\hat{q}_2;P_1,P_2\right)=&4\left(\frac{1}{\det\left(2\pi iC_{1fi}\right)\det\left(2\pi iC_{2fi}\right)}\right)^{\frac{1}{2}}\frac{1}{iD}\\&\times\exp\left[\frac{i}{2}\left\{-2D_1\hat{q}_1^2-2D_2\hat{q}_2^2+4D\hat{q}_1\hat{q}_2\right\}\right]\\&\times\exp\left[\frac{i}{2}\left\{-\frac{16}{D_1+D_2-\sqrt{(D_1+D_2)^2+D^2}}P_1^2\right.\right.\\&\left.\left.-\frac{16}{D_1+D_2+\sqrt{(D_1+D_2)^2+D^2}}P_2^2\right\}\right].\end{aligned} \tag{2.33}$$

D_1, D_2 and D read

$$D_1=\alpha_{1f}+\alpha_{1i}-2\eta_1,\quad D_2=\alpha_{2f}+\alpha_{2i}-2\eta_2\text{ and }\ D=\beta_{1f}+\beta_{1i}-2\vartheta. \tag{2.34}$$

3. Entanglement without Tunneling Effect

Wigner function is trustable method to envision analytically this process. To derive the linear entropy, the one harmonic oscillator Wigner function $W(\hat{q}_1,P_1)$ outside tunneling effect is described like

$$\begin{aligned}W\left(\hat{q}_1,P_1\right)=&\int d\hat{q}_2dP_2W\left(\hat{q}_1,\hat{q}_2;P_1,P_2\right)\\=&\frac{\sqrt{2}\pi}{D}\left(\frac{1}{\det\left(2\pi iC_{1fi}\right)\det\left(2\pi iC_{2fi}\right)}\right)^{\frac{1}{2}}\left(\frac{D_1+D_2+\sqrt{(D_1+D_2)^2+D^2}}{D_2}\right)^{\frac{1}{2}}\\&\times\exp\left[-i\frac{D_1D_2+D^2}{D_2}\hat{q}_1^2\right]\exp\left[-i\frac{8}{D_1+D_2-\sqrt{(D_1+D_2)^2+D^2}}P_1^2\right].\end{aligned} \tag{3.1}$$

Consequently, the entropy reads:

$$\begin{aligned}S=&1-(2\pi)^2\int W^2d\hat{q}_1dP_1\\=&1-\frac{\sqrt{2}}{i}\frac{\pi^5}{D^2}\left(\frac{1}{\det\left(2\pi iC_{1fi}\right)\det\left(2\pi iC_{2fi}\right)}\right)\left(\frac{D_2\left(D_1+D_2-\sqrt{(D_1+D_2)^2+D^2}\right)}{D_1D_2+D^2}\right)^{\frac{1}{2}}\\&\times\left(\frac{D_1+D_2+\sqrt{(D_1+D_2)^2+D^2}}{D_1D_2+D^2}\right).\end{aligned} \tag{3.2}$$

4. Entanglement with Tunneling Effect

The introduction mentions the importance of the tunneling effect in many fields. This work will approach this phenomenon from a new perspective, that of quantum information, in specific, entanglement.

At point A, at the meeting of the potential barrier, the one entangled harmonic oscillator is characterized in accordance to the potential energy back to (2.2) and (2.4) as

$$\frac{1}{2}\bar{\omega}_{I1}(t)\hat{Q}_1^2(t) \simeq \frac{1}{2}\bar{\omega}_{I1}(t)\hat{Q}_{A1}^2 + \frac{1}{2}\bar{\omega}_{IA1}(t)\hat{Q}_1'^2(t), \tag{4.1}$$

such that $\bar{\omega}_{IAj}(t) = \bar{\omega}_{Ij}(t)$ and $\hat{Q}_1'(t) = \hat{Q}_1 - \hat{Q}_{A1}$. We also have

$$\frac{1}{2}\xi_1(t)\hat{q}_1^2(t) \simeq \frac{1}{2}\xi_1(t)\hat{q}_{A1}^2 + \frac{1}{2}\xi_{A1}(t)\hat{q}_1'^2(t), \tag{4.2}$$

with $V(\hat{q}_1) = \frac{1}{2}\xi_1(t)\hat{q}_1^2(t)$, $\xi_{A1}(t) = \xi_1(t)$ and $\hat{q}_1'(t) = \hat{q}_1 - \hat{q}_{A1}$. Consequently

$$\hat{q}_1^2(t) \simeq \hat{q}_{A1}^2 + \hat{q}_1'^2(t), \tag{4.3}$$

Herein, we assess the Wigner function (3.1) to yield the form

$$\begin{aligned} W_A =& \frac{\sqrt{2}\pi}{D}\left(\frac{1}{\det\left(2\pi i C_{1fi}\right)\det\left(2\pi i C_{2fi}\right)}\right)^{\frac{1}{2}}\left(\frac{D_1 + D_2 + \sqrt{(D_1+D_2)^2 + D^2}}{D_2}\right)^{\frac{1}{2}} \\ &\times \exp\left[-i\frac{D_1D_2 + D^2}{D_2}\left(\hat{q}_{A1}^2 + \hat{q}_1'^2(t)\right)\right]\exp\left[-i\frac{8}{D_1 + D_2 - \sqrt{(D_1+D_2)^2+D^2}}P_1^2\right]. \end{aligned} \tag{4.4}$$

We now define the Wigner function at A as:

$$\begin{aligned} W\left(\hat{q}_{A1}, P_1\right) =& \int W_A d\hat{q}_1 \\ =& \frac{\sqrt{2}\pi}{D}\left(\frac{1}{\det\left(2\pi i C_{1fi}\right)\det\left(2\pi i C_{2fi}\right)}\right)^{\frac{1}{2}}\left(\frac{D_1 + D_2 + \sqrt{(D_1+D_2)^2 + D^2}}{D_2}\right)^{\frac{1}{2}} \\ &\times \int \exp\left[-i\frac{D_1D_2 + D^2}{D_2}\left(\hat{q}_{A1}^2 + \left(\hat{q}_1 - \hat{q}_{A1}\right)^2\right)\right]d\hat{q}_1 \\ &\times \exp\left[-i\frac{8}{D_1 + D_2 - \sqrt{(D_1+D_2)^2+D^2}}P_1^2\right]. \end{aligned} \tag{4.5}$$

A simple calculation of (4.5) gives

$$\begin{aligned} W\left(\hat{q}_{A1}, P_1\right) =& \sqrt{\frac{2\pi}{i}}\frac{\pi}{D}\left(\frac{1}{\det\left(2\pi i C_{1fi}\right)\det\left(2\pi i C_{2fi}\right)}\right)^{\frac{1}{2}}\left(\frac{D_1 + D_2 + \sqrt{(D_1+D_2)^2 + D^2}}{D_1D_2 + D^2}\right)^{\frac{1}{2}} \\ &\times \exp\left[-i\frac{D_1D_2 + D^2}{D_2}\hat{q}_{A1}^2\right]\exp\left[-i\frac{8}{D_1 + D_2 - \sqrt{(D_1+D_2)^2+D^2}}P_1^2\right]. \end{aligned} \tag{4.6}$$

The entropy at A is given following (4.6) by

$$\begin{aligned} S_A =& 1 - (2\pi)^2\int W^2 d\hat{q}_{A1}dP_1 \\ =& 1 + \frac{\pi^6}{D^2}\left(\frac{1}{\det\left(2\pi i C_{1fi}\right)\det\left(2\pi i C_{2fi}\right)}\right)\left(\frac{D_2\left(D_1 + D_2 - \sqrt{(D_1+D_2)^2 + D^2}\right)}{D_1D_2 + D^2}\right)^{\frac{1}{2}} \\ &\times \left(\frac{D_1 + D_2 + \sqrt{(D_1+D_2)^2 + D^2}}{D_1D_2 + D^2}\right). \end{aligned} \tag{4.7}$$

To cross the barrier, the potential of such a system does not cancel inside the barrier but attenuates almost exponentially to give at point C using (2.4) the following expression

$$\frac{1}{2}\bar{\omega}_{I1}(t)\hat{Q}_1^2(t) \simeq \frac{1}{2}\bar{\omega}_{I1}(t)\hat{Q}_{C1}^2 - \frac{1}{2}\bar{\omega}_{IC1}(t)\hat{Q}_1''^2(t), \tag{4.8}$$

where $\bar{\omega}_{IC1}(t) = -i\bar{\omega}_{I1}(t)$ and $\hat{Q}_1''(t) = \hat{Q}_1 - \hat{Q}_{C1}$.

Next, we back to the original variables of Hamiltonian (2.2), we have

$$\frac{1}{2}\xi_1(t)\hat{q}_1^2(t) \simeq \frac{1}{2}\xi_1(t)\hat{q}_{C1}^2 - \frac{1}{2}\xi_{C1}(t)\hat{q}_1''^2(t), \tag{4.9}$$

$\xi_{C1}(t)$ and $\hat{q}_1''(t)$ are given in the form: $\xi_{C1}(t) = -i\xi_1(t)$ and $\hat{q}_1''(t) = \hat{q}_1 - \hat{q}_{C1}$.

Here

$$\hat{q}_1^2(t) \simeq \hat{q}_{C1}^2 + i\hat{q}_1''^2(t), \tag{4.10}$$

then the Wigner function (3.1) becomes

$$\begin{aligned} W_C =& \frac{\sqrt{2}\pi}{D}\left(\frac{1}{\det(2\pi i C_{1fi})\det(2\pi i C_{2fi})}\right)^{\frac{1}{2}}\left(\frac{D_1+D_2+\sqrt{(D_1+D_2)^2+D^2}}{D_2}\right)^{\frac{1}{2}} \\ &\times \exp\left[-i\frac{D_1D_2+D^2}{D_2}\left(\hat{q}_{C1}^2 + i\hat{q}_1''^2(t)\right)\right]\exp\left[-i\frac{8}{D_1+D_2-\sqrt{(D_1+D_2)^2+D^2}}P_1^2\right]. \end{aligned} \tag{4.11}$$

At point C, the one harmonic oscillator is characterized by the Wigner function:

$$\begin{aligned} W\left(\hat{q}_{C1}, P_1\right) =& \int W_C d\hat{q}_1 \\ =& \frac{\sqrt{2}\pi}{D}\left(\frac{1}{\det(2\pi i C_{1fi})\det(2\pi i C_{2fi})}\right)^{\frac{1}{2}}\left(\frac{D_1+D_2+\sqrt{(D_1+D_2)^2+D^2}}{D_2}\right)^{\frac{1}{2}} \\ &\times \int \exp\left[-i\frac{D_1D_2+D^2}{D_2}\left(\hat{q}_{C1}^2 + i\left(\hat{q}_1 - \hat{q}_{C1}\right)^2\right)\right]d\hat{q}_1 \\ &\times \exp\left[-i\frac{8}{D_1+D_2-\sqrt{(D_1+D_2)^2+D^2}}P_1^2\right], \end{aligned} \tag{4.12}$$

and we obtain

$$\begin{aligned} W\left(\hat{q}_{C1}, P_1\right) =& \sqrt{\frac{2\pi}{i-1}}\frac{\pi}{D}\left(\frac{1}{\det(2\pi i C_{1fi})\det(2\pi i C_{2fi})}\right)^{\frac{1}{2}}\left(\frac{D_1+D_2+\sqrt{(D_1+D_2)^2+D^2}}{D_1D_2+D^2}\right)^{\frac{1}{2}} \\ &\times \exp\left[\frac{i}{i-1}\frac{D_1D_2+D^2}{D_2}\hat{q}_{C1}^2\right]\exp\left[-i\frac{8}{D_1+D_2-\sqrt{(D_1+D_2)^2+D^2}}P_1^2\right]. \end{aligned} \tag{4.13}$$

Then, based on (4.13), the corresponding entropy is given as

$$\begin{aligned} S_C =& 1-(2\pi)^2\int W^2 d\hat{q}_{C1}dP_1 \\ =& 1+\sqrt{\frac{2}{i-1}}\frac{\pi^6}{D^2}\left(\frac{1}{\det(2\pi i C_{1fi})\det(2\pi i C_{2fi})}\right)\left(\frac{D_2\left(D_1+D_2-\sqrt{(D_1+D_2)^2+D^2}\right)}{D_1D_2+D^2}\right)^{\frac{1}{2}} \\ &\times\left(\frac{D_1+D_2+\sqrt{(D_1+D_2)^2+D^2}}{D_1D_2+D^2}\right). \end{aligned} \tag{4.14}$$

5. Correction Factor

This section deals with the calculation of the correction factor. The passage of the harmonic oscillator through a potential barrier necessarily imposes a quantum correction. This correction is generally linked to the tunnel effect, where the particle is in phase transition. Chemists are very interested to compute the correction factor, particularly concerning systems in phase transition-state theory [26]–[29]. We study it in this paper for an entangled system as it passes through the potential barrier. To start, the one harmonic oscillator successfully crosses the barrier and finally reaches region B with potential energy

$$\frac{1}{2}\bar{\omega}_{I1}(t)\hat{Q}_1^2(t) \simeq \frac{1}{2}\bar{\omega}_{I1}(t)\hat{Q}_{B1}^2 - \frac{1}{2}\bar{\omega}_{IB1}(t)\hat{Q}_1''^2(t), \tag{5.1}$$

We read $\bar{\omega}_{IB1}(t)$ and $\hat{Q}_1'''(t)$ respectively as $\bar{\omega}_{IB1}(t) = \bar{\omega}_{I1}(t)$ and $\hat{Q}_1'''(t) = \hat{Q}_1 - \hat{Q}_{B1}$.
Similarly employing Hamiltonian (2.2), we have

$$\frac{1}{2}\xi_1(t)\hat{q}_1^2(t) \simeq \frac{1}{2}\xi_1(t)\hat{q}_{B1}^2 - \frac{1}{2}\xi_{B1}(t)\hat{q}_1'''^2(t). \tag{5.2}$$

$\xi_{B1}(t)$ and $\hat{q}_1'''(t)$ getting $\xi_{B1}(t) = \xi_1(t)$ and $\hat{q}_1'''(t) = \hat{q}_1 - \hat{q}_{B1}$.
Herein the coordinate $\hat{q}_1$ satisfies the condition

$$\hat{q}_1^2(t) \simeq \hat{q}_{B1}^2 - \hat{q}_1'''^2(t), \tag{5.3}$$

to obtain the Wigner function in (3.1) as

$$\begin{aligned} W_B =& \frac{\sqrt{2}\pi}{D}\left(\frac{1}{\det\left(2\pi i C_{1fi}\right)\det\left(2\pi i C_{2fi}\right)}\right)^{\frac{1}{2}}\left(\frac{D_1+D_2+\sqrt{(D_1+D_2)^2+D^2}}{D_2}\right)^{\frac{1}{2}} \\ & \times \exp\left[-i\frac{D_1D_2+D^2}{D_2}\left(\hat{q}_{B1}^2-\hat{q}_1'''(t)\right)\right]\exp\left[-i\frac{8}{D_1+D_2-\sqrt{(D_1+D_2)^2+D^2}}P_1^2\right], \end{aligned} \tag{5.4}$$

thus we obtain at region B:

$$\begin{aligned} W\left(\hat{q}_{B1},P_1\right) =& \int W_B d\hat{q}_1 \\ =& \frac{\sqrt{2}\pi}{D}\left(\frac{1}{\det\left(2\pi i C_{1fi}\right)\det\left(2\pi i C_{2fi}\right)}\right)^{\frac{1}{2}}\left(\frac{D_1+D_2+\sqrt{(D_1+D_2)^2+D^2}}{D_2}\right)^{\frac{1}{2}} \\ & \times \int \exp\left[-i\frac{D_1D_2+D^2}{D_2}\left(\hat{q}_{B1}^2-\left(\hat{q}_1-\hat{q}_{B1}\right)^2\right)\right]d\hat{q}_1 \\ & \times \exp\left[-i\frac{8}{D_1+D_2-\sqrt{(D_1+D_2)^2+D^2}}P_1^2\right]. \end{aligned} \tag{5.5}$$

Then one can find $W\left(\hat{q}_{B1},P_1\right)$, which is

$$\begin{aligned} W\left(\hat{q}_{B1},P_1\right) =& \sqrt{2i\pi}\frac{\pi}{D}\left(\frac{1}{\det\left(2\pi i C_{1fi}\right)\det\left(2\pi i C_{2fi}\right)}\right)^{\frac{1}{2}}\left(\frac{D_1+D_2+\sqrt{(D_1+D_2)^2+D^2}}{D_1D_2+D^2}\right)^{\frac{1}{2}} \\ & \times \exp\left[-\frac{D_1D_2+D^2}{D_2}\hat{q}_{B1}^2\right]\exp\left[-i\frac{8}{D_1+D_2-\sqrt{(D_1+D_2)^2+D^2}}P_1^2\right]. \end{aligned} \tag{5.6}$$

Estimate the transmission probability as the ratio between the intensity emitted at region B and the incident intensity of region A:

$$P_T = \frac{|\psi_{tran}|^2}{|\psi_{in}|^2}. \tag{5.7}$$

To start, we are interested by computing respectively transmission and incident probabilities as

$$\begin{aligned} |\psi_{tran}|^2 =& \frac{1}{2\pi}\int W\left(\hat{q}_{B1},P_1\right)dP_1 \\ =& \frac{\pi}{4D}\left(\frac{1}{\det\left(2\pi i C_{1fi}\right)\det\left(2\pi i C_{2fi}\right)}\right)^{\frac{1}{2}}\left(\frac{(D_1+D_2)^2-\left((D_1+D_2)^2+D^2\right)}{D_1D_2+D^2}\right)^{\frac{1}{2}} \\ & \times \exp\left[-\frac{D_1D_2+D^2}{D_2}\hat{q}_{B1}^2\right], \end{aligned} \tag{5.8}$$

and

$$|\psi_{in}|^2 = \frac{1}{2\pi}\int W\left(\hat{q}_{A1}, P_1\right) dP_1$$
$$= \frac{\pi}{4iD}\left(\frac{1}{\det\left(2\pi iC_{1fi}\right)\det\left(2\pi iC_{2fi}\right)}\right)^{\frac{1}{2}}\left(\frac{(D_1+D_2)^2-\left((D_1+D_2)^2+D^2\right)}{D_1D_2+D^2}\right)^{\frac{1}{2}} \tag{5.9}$$
$$\times\exp\left[-i\frac{D_1D_2+D^2}{D_2}\hat{q}_{A1}^2\right].$$

The interference effect of the coherent state of the usual states in expressions (5.8) and (5.9) give respectively:

$$|\psi_{tran}|^2 = \frac{\pi}{4D}\left(\frac{1}{\det\left(2\pi iC_{1fi}\right)\det\left(2\pi iC_{2fi}\right)}\right)^{\frac{1}{2}}\left(\frac{(D_1+D_2)^2-\left((D_1+D_2)^2+D^2\right)}{D_1D_2+D^2}\right)^{\frac{1}{2}}$$
$$\times\exp\left[-\frac{D_1D_2+D^2}{D_2}\left[\left(\hat{q}_{B1}-\beta_1\right)^2+\left(\hat{q}_{B1}+\beta_1\right)^2\right]\right], \tag{5.10}$$

$$|\psi_{in}|^2 = \frac{\pi}{4iD}\left(\frac{1}{\det\left(2\pi iC_{1fi}\right)\det\left(2\pi iC_{2fi}\right)}\right)^{\frac{1}{2}}\left(\frac{(D_1+D_2)^2-\left((D_1+D_2)^2+D^2\right)}{D_1D_2+D^2}\right)^{\frac{1}{2}}$$
$$\times\exp\left[-i\frac{D_1D_2+D^2}{D_2}\left[\left(\hat{q}_{A1}-\alpha_1\right)^2+\left(\hat{q}_{A1}+\alpha_1\right)^2\right]\right]. \tag{5.11}$$

From (5.7), results (5.8) and (5.9) give

$$P_T = i\exp\left[-\frac{D_1D_2+D^2}{D_2}\left[\left(\hat{q}_{B1}-\beta_1\right)^2-i\left(\hat{q}_{A1}-\alpha_1\right)^2+\left(\hat{q}_{B1}+\beta_1\right)^2-i\left(\hat{q}_{A1}+\alpha_1\right)^2\right]\right]. \tag{5.12}$$

Consequently, we have a correction factor equivalent of

$$\Gamma = \beta P_T\int_{E_0}^{+\infty}\exp\left[-\beta\left(E-V\right)\right]dE. \tag{5.13}$$

E is the energy of the system, β define the inverse of temperature and V the height of the barrier. The ground state impose the corresponding energy $E_0 = \frac{\bar{\omega}_{I1}(t)}{2}$, so $\int_{E_0}^{+\infty}\exp\left[-\beta\left(E-V\right)\right]dE = \frac{1}{\beta}\exp\left(-\beta\left(E_0-V\right)\right)$.

The generalization of expression (5.11) read

$$\Gamma = \exp\left(-\beta\left(E_0-V\right)\right)P_T. \tag{5.14}$$

$$\theta\left(E\right) = \int_{\hat{q}_{A1}-\alpha_1}^{\hat{q}_{B1}-\beta_1}\left[\frac{1}{2}\frac{\bar{\omega}_{I1}(t)}{\sqrt{g_{-1}(t)}}\hat{q}_1^2(t)-E\right]^{\frac{1}{2}}d\hat{q}_1+\int_{\hat{q}_{A1}+\alpha_1}^{\hat{q}_{B1}+\beta_1}\left[\frac{1}{2}\frac{\bar{\omega}_{I1}(t)}{\sqrt{g_{-1}(t)}}\hat{q}_1^2(t)-E\right]^{\frac{1}{2}}d\hat{q}_1$$
$$= \frac{1}{2}\frac{\bar{\omega}_{I1}(t)}{\sqrt{g_{-1}(t)}}\left[\frac{\hat{q}_{B1}-\beta_1}{\left(\frac{1}{2}\frac{\bar{\omega}_{I1}(t)}{\sqrt{g_{-1}(t)}}\left(\hat{q}_{B1}(t)-\beta_1\right)^2-E\right)^{\frac{1}{2}}}+\frac{\hat{q}_{B1}+\beta_1}{\left(\frac{1}{2}\frac{\bar{\omega}_{I1}(t)}{\sqrt{g_{-1}(t)}}\left(\hat{q}_{B1}(t)+\beta_1\right)^2-E\right)^{\frac{1}{2}}}\right. \tag{5.15}$$
$$\left.+\frac{\hat{q}_{B1}-\alpha_1}{\left(\frac{1}{2}\frac{\bar{\omega}_{I1}(t)}{\sqrt{g_{-1}(t)}}\left(\hat{q}_{A1}(t)-\alpha_1\right)^2-E\right)^{\frac{1}{2}}}+\frac{\hat{q}_{A1}+\alpha_1}{\left(\frac{1}{2}\frac{\bar{\omega}_{I1}(t)}{\sqrt{g_{-1}(t)}}\left(\hat{q}_{A1}(t)+\alpha_1\right)^2-E\right)^{\frac{1}{2}}}\right].$$

6. Numerical Illustration

We are interested to deal with two different model to implement the importance of the analytical results obtained in (3.2), (4.7), (4.14), and (5.12). With constant mass, time-dependent angular frequency and coupling parameter, we consider the first model of two coupled harmonic oscillators. In this context, the three parameters c_{1j}, c_{2j} and c_{3j} in (2.6) become $c_{1j} = \frac{1}{M}$, $c_{2j} = 0$ and $c_{3j} = \frac{S_j^4}{M}$. S_j are the Schrödinger-picture operators. They are defined by $S_j = S_0 e^{i\xi_j(t)}$ [30], [31].

f_{1j}, f_{2j} solutions of (2.6) become

$$f_{1j}(t) = \cos \xi_j(t), \text{ and } f_{2j}(t) = \sin \xi_j(t). \tag{6.1}$$

Consequently (2.6)–(2.8) lead to

$$g_{-j}(t) = \frac{1}{M}\left(\cos^2 \xi_j t + e^{i4\xi_j t} \sin^2 \xi_j t\right), \tag{6.2}$$

$$g_{0j}(t) = \left(e^{i4\xi_j t} - 1\right) \xi_j \cos \xi_j t \sin \xi_j t, \tag{6.3}$$

$$g_{+j}(t) = M\xi_j^4 \left(\sin^2 \xi_j t + e^{i4\xi_j t} \cos^2 \xi_j t\right), \tag{6.4}$$

and (2.5) gives

$$\bar{\omega}_{Ij}(t) = \xi_j e^{i4\xi_j t}. \tag{6.5}$$

To be more explicit, we plot in Fig. 2; entanglement entropies (3.2), (4.7), (4.14) are write respectively $S(t)$, $S_A(t)$ and $S_C(t)$, compared to different values of the coupling parameter λ and in Fig. 3, the correction factor for three values of temperature .

We notice that there is an interference between the values of entanglement in accordance with different values of the coupling parameter λ. The sinusoïdal character of functions (6.1) reinforces the oscillatory behavior in the entanglement dynamics. The fast process of these oscillations reveals multi-frequency lines. We are interested in the evolution of entanglement of a harmonic oscillator when crossing a potential barrier by a tunnel effect. In the usual state, we consider that the system is moderately entangled. Facing the potential barrier (point A), entanglement increases rapidly. Inside the potential barrier, entanglement is almost keeping its value at the entry. Very large values is considered of the correction factor and it increases with the values of the height of the potential barrier V and the β parameter. Quantum correction is an essential element of fault computing which must manage errors in the stored information. The greater the height of the potential barrier, the more disturbed the entangled harmonic oscillator. This trend is reversed by following the temperature.

The second interesting model concerns damped pulsating coupled harmonic oscillators with time-dependent mass [32]

$$M(t) = M(t_0) \exp\left[2\left(\gamma t + \mu \sin \nu t\right)\right], \tag{6.6}$$

and frequencies, following (2.3) as

$$\xi_j(t) = \sqrt{\xi_{0j}^2 + \frac{1}{M(t)} \frac{d^2\sqrt{M(t)}}{dt^2}}. \tag{6.7}$$

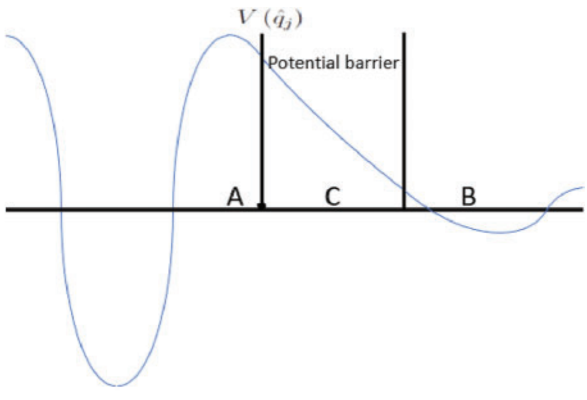

Fig. 1. Illustration of the trajectory of one entangled harmonic oscillator (blue solid line) outside and inside the potential barrier.

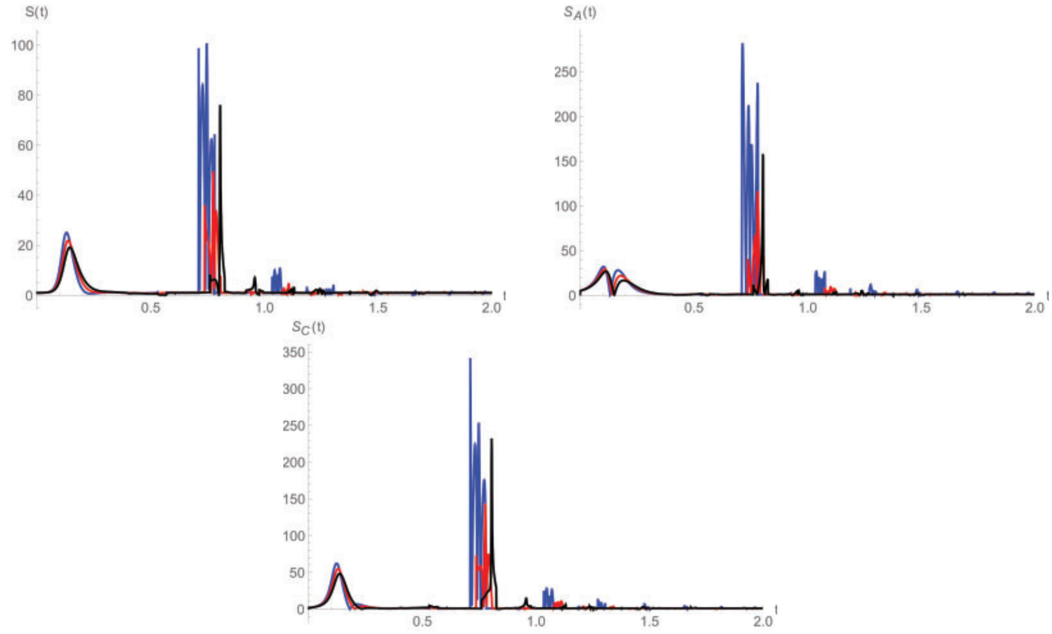

Fig. 2. Plots of (3.2), (4.7) and (4.14) for $\lambda = 0.16$ (blue solid line), $\lambda = 0.49$ (red solid line), $\lambda = 0.80$ (black solid line).

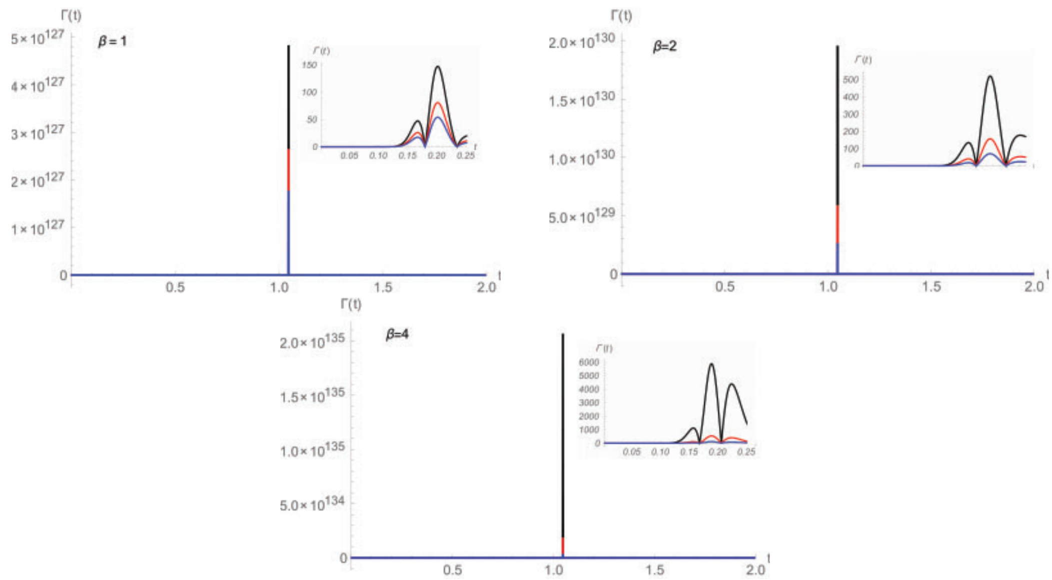

Fig. 3. Plots of (5.12) for different values of V; $V = 0.8$ (blue solid line), $V = 1.1$ (red solid line), $V = 1.8$ (black solid line).

We can establish parameters of expression (2.6) as

$$c_{1j}(t) = -M(t)\Big[\xi_j \tan\big(\xi_j t - \phi\big) + (\gamma + \mu\nu\cos\nu t)\Big], \tag{6.8}$$

$$c_{2j}(t) = -2\ln\left|\frac{\cos\big(\xi_j t - \phi\big)}{\cos\phi}\exp\Big[-(\gamma t + \mu\sin\nu t)\Big]\right|, \tag{6.9}$$

$$c_{3j}(t) = -\frac{1}{M(t_0)\xi_j}\frac{\sin\xi_j t\cos\phi}{\cos\big(\xi_j t - \phi\big)}, \tag{6.10}$$

and

$$f_j(t) = \frac{A\cos\xi_j t + B\sin\xi_j t}{\sqrt{M(t)}}, \tag{6.11}$$

with $\tan\phi = \dfrac{B}{A}$. In the specific case where the classical solutions of motion $f_j(t)$ evolve separably into $\cos\xi_j$ only, we have $A = 1$ and $B = 0$.

Expression (2.5) becomes

$$\bar{\omega}_{I1}(t) = A_1(t)\Big[\xi_1\xi_2 J_1(t)\bar{J}_1(t) - \frac{1}{2}\xi_1^2 J_2^2(t) - \frac{1}{2}\xi_2^2\bar{J}_2^2(t)\Big] + A_2(t)\Big(\xi_1 J_1(t) - \xi_2\bar{J}_1(t)\Big)^2 + A_3(t)\Big[\frac{\xi_2}{M(t_0)\xi_1}\frac{J_2^2 t}{\cos^2\xi_2 t} + J_2(t)\ln\Big|\cos\xi_1 t\exp\Big[-\Big(\gamma t + \mu\sin\eta t\Big)\Big]\Big|\Big], \tag{6.12}$$

where

$$A_1(t) = 2\ln^2\Big|\cos\xi_1 t\exp\Big[-(\gamma t + \mu\sin\eta t)\Big]\Big|, \tag{6.13}$$

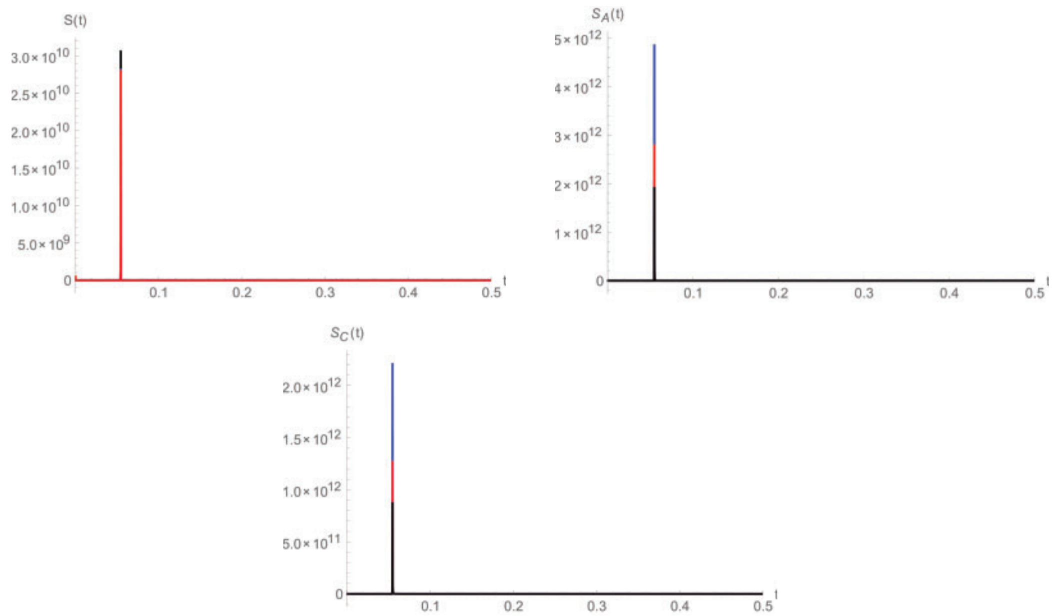

Fig. 4. Plots of (3.2), (4.7) and (4.14) for λ = 0.2 (blue solid line), λ = 0.4 (red solid line), λ = 0.6 (black solid line).

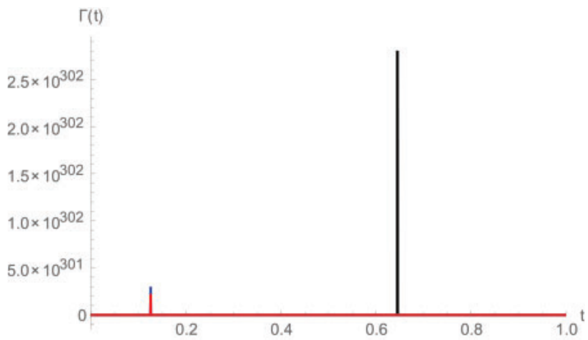

Fig. 5. Plots of (5.14) with out interference effect, for $V = 0.1$, $\beta = 0.3$ and different values of λ; λ = 0.49 (blue solid line), λ = 0.5 (red solid line), λ = 0.58 (black solid line).

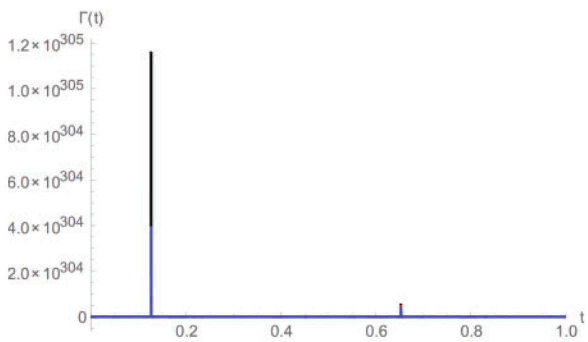

Fig. 6. Plots of (5.14), for $V = 0.1$, $\beta = 0.3$, $\lambda = 0.5$ and different values of the couple (α_1, β_1); (0.001, 0.002) (blue solid line), (0.002, 0.003) (red solid line), (0.003, 0.004) (black solid line).

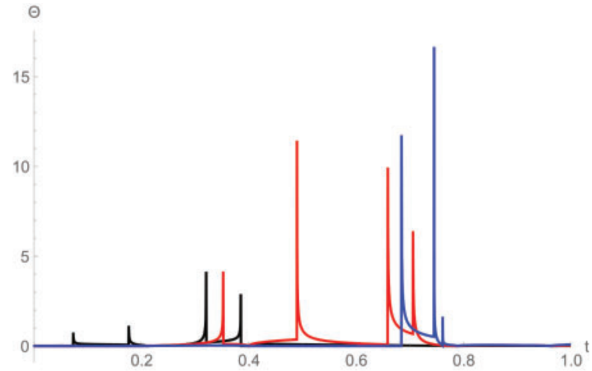

Fig. 7. Result of expression (5.15), for $V = 0.1$, $\beta = 0.3$, $\lambda = 0.5$ and different values of the couple (α_1, β_1); (0.1, 0.5) (blue solid line), (0.6, 1) (red solid line), (1.1, 1.5) (black solid line).

$$A_2(t) = \frac{M(t)}{M(t_0)\xi_1}\tan\xi_1 t\Big[\xi_1\tan\xi_1 t + (\gamma + \mu\cos\eta t)\Big], \tag{6.14}$$

$$A_3(t) = \frac{\xi_{02}}{M(t_0)\xi_{01}}\Big(\cos^2\xi_2(t) - \sin^2\xi_2(t)\Big), \tag{6.15}$$

and

$$\begin{aligned} j_1(t) &= \sin\xi_1 t\sin\xi_2 t, \quad \bar{j}_1(t) = \cos\xi_1 t\cos\xi_2 t, \\ j_2(t) &= \sin\xi_1 t\cos\xi_2 t, \text{ and } \bar{j}_2(t) = \sin\xi_2 t\cos\xi_1 t. \end{aligned} \tag{6.16}$$

In the same manner, we have

$$\begin{aligned} \bar{\omega}_{I2}(t) = A_1'(t)\Big[\xi_1\xi_2 j_1(t)\bar{j}_1(t) - \frac{1}{2}\xi_1^2 j_2^2(t) - \frac{1}{2}\xi_2^2\bar{j}_2^2(t)\Big] + A_2'(t)\Big(\xi_1 j_1(t) - \xi_2\bar{j}_1(t)\Big)^2 \\ + A_3'(t)\Big[\frac{1}{M(t_0)}\sin^2\xi_2 t + 2\xi_2\bar{j}_2 t\ln\Big|\cos\xi_1 t\exp\Big[-\Big(\gamma t + \mu\sin\eta t\Big)\Big]\Big|\Big], \end{aligned} \tag{6.17}$$

where

$$A_1'(t) = 2\ln^2\Big|\cos\xi_2 t\exp\Big[-(\gamma t + \mu\sin\eta t)\Big]\Big|, \tag{6.18}$$

$$A_2'(t) = \frac{M(t)}{M(t_0)\xi_2}\tan\xi_2 t\Big[\xi_2\tan\xi_2 t + (\gamma + \mu\cos\eta t)\Big], \tag{6.19}$$

$$A_3'(t) = \frac{1}{M(t_0)}\Big(\cos^2\xi_2 t - \sin^2\xi_2 t\Big). \tag{6.20}$$

We set $M(t_0) = 0.6$, $\gamma = 0.6$, $\mu = 1.3$, $\nu = 1.6$, $T = 1$ and the frequencies $\Omega_1 = 0.5$, $\Omega_2 = 1.1$.

Compared to the previous case, quantum entanglement is more important because of the damping effect. The rapid increase of this concept removes the oscillatory behavior and only the linear behavior is considered.

7. Conclusion

We have analytically computed the time-dependent linear entanglement entropies and the correction factor of two coupled harmonic oscillators without and across the potential barrier. Our analysis employs the Wigner function of the mixed state found at an exact treatment applying the Heisenberg picture approach. Entanglement dynamics increases upon encountering the potential barrier and its value persists inside. We illustrate the results between simple and damped harmonic oscillators. We show that entanglement becomes huge due to the damping. The higher the potential barrier and the lower the temperature, the more unstable the harmonic oscillator appears. An increase of entanglement and the application of the interference effect increases the correction factor. Interference effect also indicate an interference between the values of the barrier penetration integral.

Acknowledgment

We thank Prof. Hichem Eleuch for useful discussions.

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