Analytical Investigation of the Single-Particle Energy Spectrum in Magic Nuclei of ^{56}Ni and ^{116}Sn

J. A. Obu, E. S. William, I. O. Akpan, E. A. Thompson and E. P. Inyang

ABSTRACT

The analytical solutions of the radial D-dimensional Schrödinger equation for the Yukawa potential plus spin-orbit and Coulomb interaction terms are presented within the framework of the Nikiforov-Uvarov method by using the Greene-Aldrich approximation scheme to the centrifugal barrier. The energy eigenvalues obtained are employed to calculate the single-energy spectrum of ^{56}Ni and ^{116}Sn for distinct quantum states. We have also obtained corresponding normalized wave functions for the magic nuclei manifested in terms of Jacobi polynomials. However, the energy spectrum without Spin-orbit and Coulomb interaction terms precisely matches the quantum mechanical system of the Yukawa potential field at any arbitrary state.

Keywords: D-dimensional Schrodinger equation, Nikiforov-Uvarov method, spin-orbit and Coulomb interaction terms, Yukawa potential.

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I. INTRODUCTION

The ideal description and interpretation of the features of a many-body system can be studied considering the atomic nucleus as intricate quantum system designed by two distinct fermions, the proton, and the neutron [1], [2]. After the discovery of the atomic nucleus, its structure has been studied in detail using different nuclear models. The nuclear shell model (arrangement of protons and neutrons within a nucleus) led nuclear research to the modern model of the atomic nucleus [3], [4]. The strength of the nuclear shell model is well established for nuclei approaching the valley of stability. It effectively predicts and explains the occurrence of magic numbers [5]-[8]. Magic numbers corresponding to the filling of nuclear shells of single-particle levels exist since the middle of the past century [9]-[13]. It can be explained in terms of the shell model of the nucleus. This model uses quantum mechanics to describe the energetics of the nucleons [14] by assuming that each nucleon moves independently in the nucleus not influenced by the motion of the other nucleons. Furthermore, each nucleon moves in a potential well, which is constant from the center of the nucleus to its edge, where it increases rapidly by its energy.

The interaction between a proton and a neutron with all other nucleons generates an average potential field where each nucleon moves. The interacted nucleons have a nuclear mean-field potential that depends on the position coordinates [15]. There is a capability to calculate the mean-

field potential in nuclei. This purpose ranged from the empirical picture to fit the experimental data and to derive it on a small scale from the bare nucleon-nucleon potential. Thus, there is no unique nucleon-nucleon potential to be the start point [16], [17]. However, since the strong nuclear force is short ranged, we would expect the potential to follow the form of the density distribution of nucleons in the nucleus. Therefore, any short-range potential that would reproduce a single-particle energy spectrum must be constrained to this shape.

In quantum mechanics, Schrodinger equation (SE) is one of the central equations which draw great attention in physical sciences. Much advanced mathematical methods with specific potential have been used to solve this equation. Among the most popular methods are Nikiforov-Uvarov method NU [18]-[24], asymptotic iteration method [25]-[28], supersymmetric shape invariance approach [29]-[31], factorization method [32]. Studies show that the extension of physical problems to higher-dimensional space is of great concern [33]. Also, [34] noted that the exact solutions of both the relativistic and nonrelativistic wave equation with certain physical potential in higher dimensions are remarkably important not only in physics and chemistry but also in pure and applied mathematics.

More so, a single potential alone cannot reproduce experimentally observed magic numbers. Including the spin-orbit and Coulomb interaction terms to the mean-field potential, it is possible to reproduce these numbers [35]-[37]. The effect of spin-orbit coupling in the mean-field

arises if there is a contribution of orbits with $l \geq 0$. Such trajectories occur when the constraint on the particle reflects the nuclear radius [38]. The contribution of the Coulomb term is as a result of interactions between the protons in the nucleus.

Hence, this study aims to investigate the analytical solution of the radial D-dimensional Schrodinger equation for the Yukawa potential in the presence of spin-orbit and Coulomb interaction terms to study magic nuclides of ^{58}Ni and ^{116}Sn . The Yukawa potential [39] is a useful non-relativistic potential describing the strong interactions between nucleons [40], [41]. The potential is of the form:

$$V_Y(r) = -\frac{\eta e^{-\zeta r}}{r} \quad (1)$$

where η is the strength of the Yukawa potential, ζ is the screening parameter and r is the nuclear distance between two interacting nucleons [42]. In quantum mechanics, the physics of this potential depends strongly on the value of the screening parameter and also on the type of the problem under consideration.

II. REVIEW OF NIKIFOROV-UVAROV (NU) METHOD

We briefly introduce the NU method in this section; the details can be found in ref. [18]. This method is based on solving the second-order differential equation of the hypergeometric type:

$$\psi''(z) + \frac{\tilde{\tau}(z)}{\sigma(z)}\psi'(z) + \frac{\tilde{\sigma}(z)}{\sigma^2(z)}\psi(z) = 0 \quad (2)$$

where $\sigma(z)$ and $\tilde{\sigma}(z)$ are polynomials at most second-degree, $\tilde{\tau}(z)$ is a first-degree polynomial [43]. The particular solution of (2) is found by decomposing the wave function $\psi(z)$ as:

$$\psi(z) = \phi(z)y(z) \quad (3)$$

which result in a hypergeometric-type equation of the form:

$$\sigma(z)y''(z) + \tau(z)y'(z) + \lambda y(z) = 0. \quad (4)$$

The first part of (3), $\phi(z)$ is the solution of the differential equation:

$$\sigma(z)\phi'(z) - \pi(z)\phi(z) = 0 \quad (5)$$

where

$$\tau(z) = \tilde{\tau}(z) + 2\pi(z) \quad (6)$$

and λ in (4) is define as:

$$\lambda = \lambda_n = -n\tau'(s) + \frac{n(n-1)}{2}\sigma''(s) = 0, (n = 0, 1, 2, \dots) \quad (7)$$

The $\tau(z)$ is a polynomial and the first derivative $\tau'(z)$ must be negative for proper solutions. The second part in (3) is a hypergeometric-type function obtained by Rodrigues relation:

$$y_n(z) = \frac{B_n(z)}{\rho(z)} \frac{d^n}{dz^n} [\sigma^n(z)\rho(z)] \quad (8)$$

We define B_n as a constant related to normalization and $\rho(z)$ is of the form:

$$\frac{d}{dz} [\sigma(z)\rho(z)] = \tau(z)\rho(z) \quad (9)$$

We also define the function $\pi(z)$ and the parameter λ as:

$$\pi(z) = \frac{\sigma'(z) - \tilde{\tau}(z)}{2} \pm \sqrt{\left(\frac{\sigma'(z) - \tilde{\tau}(z)}{2}\right)^2 - \tilde{\sigma}(z) + k\sigma(z)} \quad (10)$$

Where $\pi(z)$, is a polynomial of first degree, and

$$\lambda = k + \pi'(z) \quad (11)$$

To obtain the value of k , the term under the square root sign in (10) is set to zero [43]. Subsequently, by solving (7) and (11), we obtained the energy eigenvalue equation.

III. SOLUTION OF THE SCHRODINGER EQUATION WITH YUKAWA POTENTIAL PLUS SPIN-ORBIT AND COULOMB INTERACTION TERMS

The Schrodinger equation in D – dimensional space of nucleons which interacts with each other is of the form [43]:

$$\frac{\hbar^2}{2\mu} \left[\frac{d^2}{dr^2} + \frac{D-1}{r} \frac{d}{dr} - \frac{\ell(\ell+D-2)}{r^2} \right] R_{n\ell j}(r) = (E_{n\ell j} - V(r)) R_{n\ell j}(r) \quad (12)$$

where $D = 3N - 3$. Here, D is the space dimension, N is the number of particles, μ is the reduced mass of a single nucleus, $E_{n\ell j}$ is the energy spectra, $\hbar$ is the reduced Planck's constant and n, ℓ and j are the principal, orbital and total angular momentum quantum numbers respectively. By using of the change of variable $\varphi_{n\ell j}(r) = r R_{n\ell j}(r)$, we have:

$$\varphi_{n\ell j}(r) = R_{n\ell j}(r) + r \frac{R_{n\ell j}(r)}{dr}. \quad (13)$$

For two particles system (i. e. proton and neutron where $N = 2$ and $D = 3$), (12) reduces to:

$$\frac{d^2 \varphi_{nlj}(r)}{dr^2} + \frac{2\mu}{\hbar^2} \left[E_{nlj} - V(r) - \frac{\hbar^2 \ell(\ell+1)}{2\mu r^2} \right] \varphi_{nlj}(r) = 0, \quad (14)$$

The spin-orbit term of the central Yukawa potential is of the form:

$$V(r)_{(so)}(r) = \frac{1}{2} \eta_{(so)} \left(\frac{r_0}{\hbar} \right)^2 \frac{1}{r} \left[\frac{d}{dr} \frac{e^{-\zeta r}}{r} \right] < \vec{\ell}, \vec{s} > \quad (15)$$

where the spin-orbit coupling constant, $\eta_{(so)} = 0.44\eta$. For spin half particle, the spin-orbit coupling term, $< \vec{\ell}, \vec{s} > = \frac{1}{2} \hbar^2 \left(j(j+1) - l(l+1) - \frac{3}{4} \right)$. In addition to the mean-field plus spin-orbit interaction, protons in nuclei interact together via the Coulomb force, which is defined by the following relation, considering nuclei as a sphere with a constant charge density [44]:

$$V_c(r) = \begin{cases} \frac{e^2}{R_c} \left[3 - \left(\frac{r}{R_c} \right)^2 \right] & r \leq R_c, \\ \frac{8e^2}{r}, & r \geq R_c, \end{cases} \quad (16)$$

the effective potential is given as the combination of (1), (15) and (16) for $r < R_c$:

$$V(r) = -\frac{\eta e^{-\zeta r}}{r} + \frac{1}{2} \eta_{(so)} r_0^2 \frac{1}{r} \left[\frac{d}{dr} \frac{e^{-\zeta r}}{r} \right] \left(j(j+1) - l(l+1) - \frac{3}{4} \right) + \left(\frac{3e^2}{\pi \epsilon_0 R_c} - \frac{e^2 r^2}{\pi \epsilon_0 R_c^3} \right) \quad (17)$$

where R_c is the Coulomb radius, e is the electronic charge and ϵ_0 is the medium of the particles. By substituting (17) into (14), we have:

$$\frac{d^2 \varphi(r)}{dr^2} + \frac{2\mu}{\hbar^2} \left[E_{nlj} + \frac{\eta e^{-\zeta r}}{r} + \frac{1}{2} \eta_{(so)} r_0^2 \frac{1}{r} \left(\zeta \frac{e^{-\zeta r}}{r} + \frac{e^{-\zeta r}}{r^2} \right) \left(j(j+1) - l(l+1) - \frac{3}{4} \right) - \left(\frac{3e^2}{\pi \epsilon_0 R_c} - \frac{e^2 r^2}{\pi \epsilon_0 R_c^3} \right) - \frac{\hbar^2}{2\mu r^2} l(l+1) \right] \varphi(r) = 0 \quad (18)$$

To get the solution of (18), we need another change of variable with:

$$z = e^{-\zeta r} \quad (19)$$

To deal with the centrifugal barrier, we introduce the Greene-Aldrich approximation scheme [45]-[47] to analytically solve the equation. This approximation scheme is an excellent approximation to the centrifugal term, which is valid for $\zeta \ll 1$. It has the form

$$\frac{1}{r^2} \approx \frac{\zeta^2}{(1 - e^{-\zeta r})^2} \quad (20)$$

By using (19), (20), and the change of variable in (19), the Schrodinger equation reads

$$\left[E_{nlj} + \frac{\eta \zeta z}{(1-z)} + \frac{1}{2} \eta_{(so)} r_0^2 \zeta^3 \left(\frac{z}{(1-z)^2} + \frac{z}{(1-z)^3} \right) - \left(\frac{j(j+1) - l(l+1) - \frac{3}{4}}{\pi \epsilon_0 R_c} - \frac{e^2 (1-z)^2}{\pi \epsilon_0 R_c^3} \right) - \frac{\hbar^2}{2\mu (1-z)^2} l(l+1) \right] \varphi_{nlj}(z) = 0 \quad (21)$$

To reach an approximate solution in (21), we apply the principle of [48] to higher degree polynomials greater than two. Hence,

$$\left[\frac{2\mu E_{nlj}}{\zeta^2 \hbar^2} - \frac{6\mu e^2}{\zeta^2 \hbar^2 \pi \epsilon_0 R_c} + \frac{2\mu e^2}{\zeta^4 \hbar^2 \pi \epsilon_0 R_c^3} - \frac{l(l+1)}{\zeta^2 \hbar^2 \pi \epsilon_0 R_c} + \frac{4\mu E_{nlj} z}{\zeta^2 \hbar^2} + \frac{2\mu \eta z}{\zeta \hbar^2} + \frac{2\mu \eta_{(so)} r_0^2 \zeta}{\hbar^2} \left(j(j+1) - l(l+1) - \frac{3}{4} \right) z - \frac{8\mu e^2 z}{\zeta^4 \hbar^2 \pi \epsilon_0 R_c^3} + \frac{2\mu E_{nlj} z^2}{\zeta^2 \hbar^2} - \frac{2\mu \eta z^2}{\zeta \hbar^2} + \frac{\mu \eta_{(so)} r_0^2 \zeta}{\hbar^2} \left(j(j+1) - l(l+1) - \frac{3}{4} \right) z^2 - \frac{6\mu e^2 z^2}{\zeta^2 \hbar^2 \pi \epsilon_0 R_c} + \frac{12\mu e^2 z^2}{\zeta^4 \hbar^2 \pi \epsilon_0 R_c^3} \right] \varphi(z) = 0, \quad (22)$$

where,

$$-\chi^2 = \frac{2\mu E_{nlj}}{\zeta^2 \hbar^2} - \frac{6\mu e^2}{\zeta^2 \hbar^2 \pi \epsilon_0 R_c} + \frac{2\mu e^2}{\zeta^4 \hbar^2 \pi \epsilon_0 R_c^3} - l(l+1) \quad (23)$$

$$\nu = \frac{12\mu e^2}{\zeta^2 \hbar^2 \pi \epsilon_0 R_c} - \frac{4\mu E_{nlj}}{\zeta^2 \hbar^2} + \frac{2\mu \eta}{\zeta \hbar^2} + \frac{2\mu \eta_{(so)} r_0^2 \zeta}{\hbar^2} \left(j(j+1) - l(l+1) - \frac{3}{4} \right) - \frac{8\mu e^2}{\zeta^4 \hbar^2 \pi \epsilon_0 R_c^3} \quad (24)$$

$$-\delta = \frac{2\mu E_{nlj}}{\zeta^2 \hbar^2} - \frac{2\mu \eta}{\zeta \hbar^2} + \frac{\mu \eta_{(so)} r_0^2 \zeta}{\hbar^2} \left(j(j+1) - l(l+1) - \frac{3}{4} \right) - \frac{6\mu e^2}{\zeta^2 \hbar^2 \pi \epsilon_0 R_c} + \frac{12\mu e^2}{\zeta^4 \hbar^2 \pi \epsilon_0 R_c^3} \quad (25)$$

By substituting (23), (24) and (25) into (22) we have:

$$\frac{d^2 \varphi_{nlj}(z)}{dz} + \frac{(1-z)}{z(1-z)} \frac{d\varphi_{nlj}(z)}{dz} + \frac{1}{z^2(1-z)^2} [-\delta z^2 + \nu z - \chi^2] \varphi_{nlj}(z) = 0. \quad (26)$$

After comparing (26) with (2), we have the appropriate polynomial as:

$$\begin{aligned} \tilde{\tau}(z) &= 1 - z, & \sigma(z) &= z(1 - z), \\ \sigma'(z) &= 1 - 2z, & \sigma''(z) &= -2, & \tilde{\sigma}(z) &= -\delta z + \nu z - \chi^2. \end{aligned} \quad (27)$$

Upon inserting the polynomials given by (27) into (10) gives the polynomial:

$$\pi(z) = -\frac{z}{2} \pm \sqrt{(\Upsilon - k)z^2 + (k - \nu)z + \chi^2} \quad (28)$$

where

$$\Upsilon = \frac{1}{4} + \delta \quad (29)$$

Based on the NU method, the quadratic form under the square root sign of (28) must be solved if we set the discriminant of this quadratic equation equal to zero: $\Delta = b^2 - 4ac = 0$. This discriminant gives a new quadratic equation, which can be solved for the constant k to get the two roots. Here, we take the negative root given as

$$K = \nu - 2\chi \left[\chi - \sqrt{\Upsilon - \nu + \chi^2} \right] \quad (30)$$

$\tau(z)$ given in (6) has a negative derivative expression for this value of $\pi(z)$ [18] using (30), the most suitable expression of $\pi(z)$ is chosen as:

$$\pi(z) = -\frac{z}{2} + \chi - z \left[\chi + \sqrt{\Upsilon - \nu + \chi^2} \right] \quad (31)$$

Hence, $\tau(z)$ and $\tau'(z)$ are obtain as:

$$\begin{aligned} \tau(z) &= 1 + 2\chi - 2z \left[1 + \chi + \sqrt{\Upsilon - \nu + \chi^2} \right] \\ \tau'(z) &= -2 - 2\chi - 2\sqrt{\Upsilon - \nu + \chi^2} \end{aligned} \quad (32)$$

using (27), (31) and (32), the following expressions for λ and λ_n are obtained explicitly:

$$\lambda = -\frac{1}{2} - \sqrt{\Upsilon - \nu + \chi^2} + \nu - \chi \left[1 - 2\chi - 2\sqrt{\Upsilon - \nu + \chi^2} \right] \quad (33)$$

$$\lambda_n = n^2 + n + 2n \left[\chi + \sqrt{\Upsilon - \nu + \chi^2} \right] \quad (34)$$

By comparing (33) and (34), we obtain the dimensionless parameter χ as:

$$\chi = -\frac{1}{2} \left[\frac{\left(n + \frac{1}{2} \right)^2 + 2 \left(n + \frac{1}{2} \right) \sqrt{\frac{1}{4} + \delta - \nu + \chi^2} + \frac{1}{4} + 2\chi^2 - \nu}{\left(n + \frac{1}{2} + \sqrt{\frac{1}{4} + \delta - \nu + \chi^2} \right)} \right] \quad (35)$$

With the help of (23)-(25), we obtain bound state energy eigenvalues of Yukawa potential plus spin-orbit coupling and Coulomb interaction term as a function of quantum numbers n , l , and j as:

$$E_{nlj} = \frac{\hbar^2 \zeta^{-2} l(l+1)}{2\mu} + \frac{3\mu e^2}{\pi \epsilon_0 R_c} - \frac{e^2}{\zeta^2 \pi \epsilon_0 R_c^3} - \frac{\hbar^2 \zeta^{-2}}{8\mu} \times \left[\left(n + \frac{1}{2} \right)^2 + 2 \left(n + \frac{1}{2} \right) \sqrt{\frac{1}{4} + l(l+1) - \frac{3\mu \eta_{(so)} r_o^2 \zeta^2 \left(j(j+1) - l(l+1) - \frac{3}{4} \right)}{\hbar^2} - \frac{6\mu e^2}{\zeta^4 \hbar^2 \pi \epsilon_0 R_c^3}} \right. \\ \left. + \frac{1}{4} + 2l(l+1) - \frac{2\mu \eta}{\zeta \hbar^2} - \frac{2\mu \eta_{(so)} r_o^2 \zeta^2 \left(j(j+1) - l(l+1) - \frac{3}{4} \right)}{\hbar^2} + \frac{4\mu e^2}{\zeta^4 \hbar^2 \pi \epsilon_0 R_c^3} \right] \\ \left[n + \frac{1}{2} + \sqrt{\frac{1}{4} + l(l+1) - \frac{3\mu \eta_{(so)} r_o^2 \zeta^2 \left(j(j+1) - l(l+1) - \frac{3}{4} \right)}{\hbar^2} - \frac{6\mu e^2}{\zeta^4 \hbar^2 \pi \epsilon_0 R_c^3}} \right] \quad (36)$$

Next is to find the wave function. Using (9), the weight function $\rho(z)$ is of the form:

$$\rho(z) = z^{2\chi} (1-z)^{2\sqrt{\frac{1}{4} + \delta - \nu + \chi^2}} \quad (37)$$

and when (37) is used in (8), the Rodrigues relation is found:

$$y_n = B_n z^{-2\chi} (1-z)^{-2\sqrt{\frac{1}{4} + \gamma - \beta + \chi^2}} \frac{d^n}{dz^n} \left[z^{n+2\chi} (1-z)^{n+2\sqrt{\frac{1}{4} + \gamma - \beta + \chi^2}} \right] \quad (38) \\ \equiv P_n^{(2\epsilon, 2\sqrt{\frac{1}{4} + \gamma - \beta + \chi^2})} (1-z),$$

where P_n is the Jacobi polynomial. From Eq. (5):

$$\phi(z) = \nu^z (1-z)^{\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \delta - \nu + \chi^2} \right)} \quad (39)$$

is obtained and the wavefunction becomes:

$$\psi_{nl}(z) = N_{nl} z^\chi (1-z)^{\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \delta - \nu + \chi^2} \right)} P_n^{(2\chi, \frac{1}{2} + \sqrt{\frac{1}{4} + \delta - \nu + \chi^2})} (1-2z) \quad (40)$$

Where N_{nl} is the normalization constant. Using the normalization condition, we obtain the normalization constant as follows:

$$\frac{N_{nl}^2}{\zeta} \int_{-1}^1 \left(\frac{1-\gamma}{2} \right)^{2\chi} \left(\frac{1+\gamma}{2} \right)^\beta \left[P_n^{(2\chi, 2\beta-1)}(\gamma) \right]^2 d\gamma = 1 \quad (41)$$

Equation (41) is obtained by setting:

$$\begin{cases} \beta = 1 + \sqrt{1 + 4\delta - 4\nu + 4\chi^2}, \\ \beta - 1 = \sqrt{1 + 4\delta - 4\nu + 4\chi^2} \end{cases} \quad (42)$$

Hence by comparing (41) with standard integral of the form of (37) of [59]:

$$\int_{-1}^1 \left(\frac{1-p}{2} \right)^x \left(\frac{1+y}{2} \right)^y \left(P_n^{(2x, 2y-1)}(p) \right)^2 dp \\ = \frac{2\Gamma(x+n+1)\Gamma(y+n+1)}{n!x\Gamma(x+y+n+1)} \quad (43)$$

we obtain the normalization constant as:

$$N_{nlj} = \left[\frac{n! \chi \zeta \Gamma(2\chi + \beta + n + 1)}{\Gamma(2\chi + n + 1) \Gamma(\beta + n + 1)} \right]^{\frac{1}{2}} \quad (44)$$

IV. SPECIAL CASES

However, no experimental evidence for the solution of (36) has been recorded to the best of our knowledge. Therefore, our calculations are only for academic purposes; hence, the need to test for the accuracy of our calculation. To do this, we consider the following cases.

A. Case 1

If we set $R_c = 0$ in (17), (36) yield energy eigenvalue of Yukawa potential with spin-orbit coupling term:

$$E_{nlj} = \frac{\hbar^2 \zeta^2 l(l+1)}{2\mu} - \left[\left(n + \frac{1}{2} \right)^2 + 2 \left(n + \frac{1}{2} \right) \sqrt{\frac{1}{4} + l(l+1) - \frac{3\mu\eta_{(so)} r_o^2 \zeta \left(j(j+1) - l(l+1) - \frac{3}{4} \right)}{\hbar^2}} \right] \cdot \frac{\hbar^2 \zeta^2}{8\mu} \cdot \frac{-\frac{2\mu\eta_{(so)} r_o^2 \zeta \left(j(j+1) - l(l+1) - \frac{3}{4} \right)}{\hbar^2}}{\left(n + \frac{1}{2} + \sqrt{\frac{1}{4} + l(l+1) - \frac{3\mu\eta_{(so)} r_o^2 \zeta \left(j(j+1) - l(l+1) - \frac{3}{4} \right)}{\hbar^2}} \right)^2} \quad (45)$$

Equation (45) is similar to (27) of [35].

B. Case 2

If we set $\eta_{so} = R_c = 0$ in (17), we have energy eigenvalue of Yukawa potential as:

$$E_{nlj} = \frac{\hbar^2 \zeta^2 l(l+1)}{2\mu} - \frac{\hbar^2 \zeta^2}{8\mu} \left[\frac{(n+l+1)^2 + l(l+1) - \frac{2\mu\eta}{\zeta \hbar^2}}{(n+l+1)} \right] \quad (46)$$

Equation (46) is similar to (34) of [49]; (19) of [50]; (2) of [51]; (52) of [52]; (75) of [53]; and (27) of [54].

V. DISCUSSION

Table 1 shows the numerical single-particle energy spectrum of the doubly magic nuclide $^{56}_{28}\text{Ni}$ ($Z = 28$, $N = 28$), and a single magic nuclide $^{116}_{50}\text{Sn}$ ($Z = 50$, $N = 66$) respectively. For the spin-orbit interaction, the spin orientation degeneracy is removed in the main spin-orbit levels, causing splitting of energy levels. By Including total angular momentum quantum number j , the numerical energy spectrum was obtained at $1p_{1/2}$, $1d_{3/2}$ and $2s_{1/2}$ for $^{56}_{28}\text{Ni}$ and $1p_{1/2}$, $1d_{3/2}$, $2s_{1/2}$, $1f_{5/2}$ and $2p_{1/2}$ for $^{116}_{50}\text{Sn}$, respectively. We observed that the larger the orbital angular momentum quantum numbers ℓ , the larger the splitting. For

$\ell = 0$, only $j = \frac{1}{2}$ is possible, hence no splitting and we have the ground state of the two magic nuclei. This is indicated by $2s_{1/2}$ state.

To impose the electromagnetic interaction on the single-proton levels, one may assume that each proton lies in the repulsive Coulomb potential well, made by the other protons. To correct single-proton levels, we understood that single point-like proton lies in a uniformly charged spherical potential well with a radius. In contrast, these corrections do not lead to a significant change in the energy levels of the nuclei [55]. However, the energy distributions of nucleons are affected by the strong short-range correlations of nucleons in finite nuclei. It becomes essential when we calculate the nucleon single-particle energies and the choice of the potential parameters. Brown [56] suggested that one could introduce up to six parameters for the spin-orbit interaction for the best fit of nuclear single-particle energies. Unfortunately, the available experimental information on the single-particle energies of the deep bound states is scanty [57].

TABLE 1: SINGLE-ENERGY SPECTRUM OF ^{56}Ni AND ^{116}Sn NUCLIDES FOR:
 $R = r_o A^{1/3}$, $\eta = 40 + 0.13A$, $\zeta = 1.54 \text{ fm}$, $\hbar = 6.582850502 \times 10^{-22} \text{ MeV.s}$,
 $e = 1.19999 (\text{MeV.fm})^{1/2}$, $\eta_{(so)} = 0.44\eta$

Nuclide	j	J^π	$\eta (\text{MeV})$	$E_{nlj} (\text{MeV})$
^{56}Ni	$1p_{1/2}$	0^+	47.28	-40.204732939
	$1d_{3/2}$			-62.01202362
	$2s_{1/2}$			-127.4261691
^{116}Sn	$1p_{1/2}$	0^+	55.08	-75.29801832
	$1d_{3/2}$			-114.2349840
	$2s_{1/2}$			-231.0459655
	$1f_{5/2}$			-153.1719709
	$2p_{1/2}$			-464.6679917

VI. CONCLUSION

In this study, we obtained the approximate analytical solutions of the Schrödinger equation for the Yukawa potential plus spin-orbit and Coulomb interaction terms by approximation to the centrifugal barrier using the NU method. Explicitly, we have computed the single-energy spectrum at different states of the two magic nuclei using energy eigenvalues obtained. We have also obtained the corresponding normalized wave functions for the magic nuclei presented in terms of Jacobi polynomial. Therefore, the proposed model can also be used to investigate other magic nuclei.

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