The Euler-Lagrange Equations in Rotating Frames

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***Abstract* — We are interested in deriving the Euler-Lagrange equations of motion in rotating frames of reference since these are the real-life conditions encountered in every day. Our paper is divided into two main sections, the first section deals with centrally rotating frame, the second section deals with the peripherally rotating frame of reference in the relativistic regime of speeds.**

***Keywords* —** Euler-Lagrange Equations, General Coordinate Transformations, Relativistic Regime of Speeds, Uniform Rotation Motion.

I. Introduction

Usually, experiments occurring in rotating frames are explained from the perspective of an external, inertial frame. In the present paper, we will construct a straightforward explanation in rotating frame by applying the formalisms developed in previous work [1]-[7],[10].

II. Derivation of the General Form of the Doppler Effect Formulas

In an inertial frame K the coordinates are (T, R, Φ, Z). In a frame K' rotating with respect the inertial frame, the coordinates are (t, r, ϕ, z). The angular speed of rotation between the two frames is Ω. The transformation between the frames is [8] given in (1).

$$\begin{aligned} T &= t \\ R &= r \\ \Phi &= \phi + \Omega t \\ Z &= z \end{aligned} \tag{1}$$

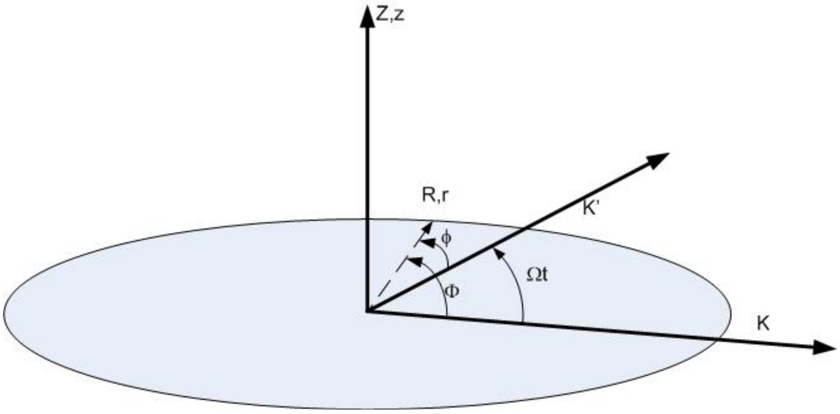

Fig. 1. Centrally rotating frame of reference.

The metric in the inertial frame is (2).

$$dS^2 = c^2 dT^2 - (dR^2 + R^2 d\Phi^2 + dZ^2) \tag{2}$$

The positive term represents the time displacement, and the negative term represents the total distance displacement. The line element expressed in the rotating frame coordinates according to (1) is (3).

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$$dS^2 = (c^2 - \Omega^2 r^2)dt^2 - dr^2 - 2\Omega r^2 d\phi dt - r^2 d\phi^2 - dz^2 \tag{3}$$

The Lagrangian is (4).

$$L = (c^2 - \Omega^2 r^2)\frac{dt^2}{dS^2} - \frac{dr^2}{dS^2} - 2\Omega r^2 \frac{d\phi}{dS}\frac{dt}{dS} - r^2\frac{d\phi^2}{dS^2} - \frac{dz^2}{dS^2} \tag{4}$$

For planar motion $\frac{d_z}{ds} = 0$ so the Lagrangian simplifies to (5).

$$\begin{aligned} L &= (c^2 - \Omega^2 r^2)\frac{dt^2}{dS^2} - \frac{dr^2}{dS^2} - 2\Omega r^2 \frac{d\phi}{dS}\frac{dt}{dS} - r^2\frac{d\phi^2}{dS^2} = \\ &= (c^2 - \Omega^2 r^2)\dot{t}^2 - \dot{r}^2 - 2\Omega r^2 \dot{\phi}\dot{t} - r^2\dot{\phi}^2 \end{aligned} \tag{5}$$

From the Lagrangian we obtain the Euler-Lagrange system of equations [1]-[5] given in (6).

$$\begin{aligned} &\frac{d}{dS}\left(\frac{\partial L}{\partial \dot{t}}\right) - \frac{\partial L}{\partial t} = 0 \\ &\frac{d}{dS}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \\ &\frac{d}{dS}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0 \end{aligned} \tag{6}$$

$$\begin{aligned} k &= \frac{\partial L}{\partial \dot{t}} = (c^2 - \Omega^2 r^2)\dot{t} - \Omega r^2 \dot{\phi} = c^2 \dot{t} + \Omega m \\ m &= \frac{\partial L}{\partial \dot{\phi}} = r^2(\Omega \dot{t} + \dot{\phi}) \\ 0 &= \frac{d}{dS}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \ddot{r} - r(\Omega^2 \dot{t}^2 + \dot{\phi}^2) \end{aligned} \tag{7}$$

The solution of the first equation is: $t = a_1 S$
Inserting into the second and third equation, (8)-(10).

$$\begin{aligned} m &= r^2(a_1\Omega + \dot{\phi}) \\ 0 &= \ddot{r} - r(a_1^2\Omega^2 + \dot{\phi}^2) \end{aligned} \tag{8}$$

$$\dot{\phi} = \frac{m}{r^2} - a_1\Omega \tag{9}$$

$$\ddot{r} - 2a_1^2\Omega^2 r + 2a_1\Omega\frac{m}{r} - \frac{m^2}{r^3} = 0 \tag{10}$$

Equations (8)-(10) do not have a symbolic solution. Not all is lost, we can try the change of variable given in (11).

$$d\tau = \sqrt{1 - \frac{r^2\Omega^2}{c^2}}\left(dt - \frac{r^2\Omega d\phi}{c^2 - r^2\Omega^2}\right) \tag{11}$$

The metric becomes (12).

$$dS^2 = c^2 d\tau^2 - dr^2 - \frac{r^2 d\phi^2}{1 - \frac{r^2\Omega^2}{c^2}} - dz^2 \tag{12}$$

The Lagrangian becomes:

$$L = c^2 \frac{d\tau^2}{dS^2} - \frac{dr^2}{dS^2} - \frac{r^2}{1-\frac{r^2\Omega^2}{c^2}} \frac{d\phi^2}{dS^2} - \frac{dz^2}{dS^2} \tag{13}$$

The Euler-Lagrange equation in r becomes:

$$0 = \frac{d}{dS}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = -2\left(\ddot{r} - \frac{r\dot{\phi}^2}{\left(1-\frac{r^2\Omega^2}{c^2}\right)^2}\right) \tag{14}$$

The Euler-Lagrange equation in ϕ is (15).

$$\ddot{\phi} = 0 \tag{15}$$

With the solution (16).

$$\dot{\phi} = m \tag{16}$$

Thus, the Euler-Lagrange equation in r becomes (17).

$$\ddot{r} = \frac{rm^2}{\left(1-\frac{r^2\Omega^2}{c^2}\right)^2} \tag{17}$$

One more variable change gives (18).

$$\frac{r\Omega}{c} = \sin a \tag{18}$$

produces (19).

$$\begin{gathered} \ddot{a}\cos^3 a - \dot{a}^2 \sin a \cos^2 a = \sin a \\ \ddot{a}\cos a - \dot{a}^2 \sin a - \sin^2 a(\ddot{a}\cos a + \dot{a}^2 \sin a) = \sin a \end{gathered} \tag{19}$$

Neglecting the term in $\sin^2 a$ we obtain the simpler equation given as (20).

$$\ddot{a}\cos a - \dot{a}^2 \sin a = \sin a \tag{20}$$

The above has an implicit solution of (21).

$$c_1 + S = \pm \frac{\sec a \sqrt{-2e^{2c_1} + \cos 2a + \tan^{-1} \frac{\sqrt{2}\sin a}{\sqrt{-2e^{2c_1} + \cos 2a + 1}}}}{\sqrt{2e^{2c_1}\sec^2 a - 2}} \tag{21}$$

Notice the similarity between (21) and (15) in reference [9]. This should come as no surprise given the similarities between gravitational fields and rotating systems.

III. The Euler-Lagrange Equations in a Peripherally Rotating Frame

The standard transformations from the peripherally rotating frame (see Fig.2) into an inertial frame, in the relativistic regime is given by [8] (22).

$$\begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} = \begin{bmatrix} \cos\alpha\cos\beta+\gamma\sin\alpha\sin\beta & \sin\alpha\cos\beta-\gamma\cos\alpha\sin\beta & 0 & -\frac{u\gamma}{c}\sin\beta \\ \cos\alpha\sin\beta-\gamma\sin\alpha\cos\beta & \sin\alpha\sin\beta+\gamma\cos\alpha\cos\beta & 0 & \frac{u\gamma}{c}\cos\beta \\ 0 & 0 & 1 & 0 \\ \frac{u\gamma}{c}\sin\alpha & -\frac{u\gamma}{c}\cos\alpha & 0 & \gamma \end{bmatrix} \begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} \quad (22)$$

$$\begin{aligned} \gamma &= \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \\ u &= r\omega \\ \alpha &= \omega\gamma\tau \\ \beta &= \omega\gamma^2\tau \end{aligned} \quad (23)$$

In the inertial frame the metric is given in (24).

$$ds^2 = (cdt)^2 - (dx^2 + dy^2 + dz^2) \quad (24)$$

Substituting (22) into (24) we obtain the metric in the rotating frame:

$$ds^2 = c^2 dt'^2 - dx'^2 - dy'^2 - dz'^2 + 4u\gamma^2 \sin\alpha\, dx' dt' - 4u\gamma^2 \cos\alpha\, dy' dt' \quad (25)$$

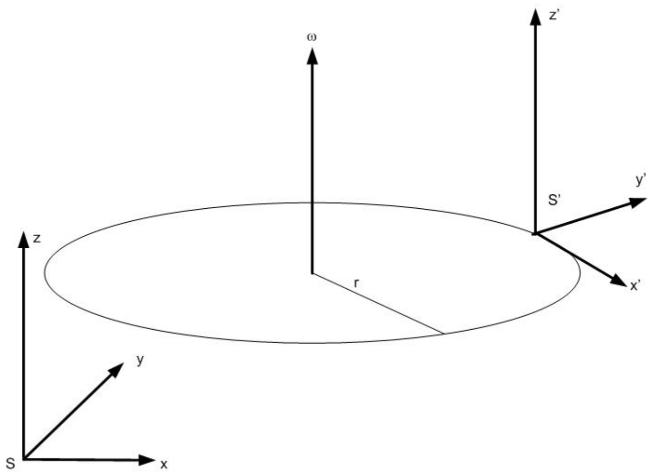

Fig. 2. Peripherally rotating frame of reference.

The Lagrangian in the rotating frame is given in (30).

$$L = c^2(\dot{t}')^2 - (\dot{x}')^2 - (\dot{y}')^2 - (\dot{z}')^2 + 4u\gamma^2 \sin\alpha\, \dot{x}'\dot{t}' - 4u\gamma^2 \cos\alpha\, \dot{y}'\dot{t}' \quad (30)$$

The Euler-Lagrange equations in the rotating frame are given in (31).

$$\begin{aligned} \frac{d}{dS}\left(\frac{\partial L}{\partial \dot{t}}\right) - \frac{\partial L}{\partial t} &= 0 \\ \frac{d}{dS}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} &= 0 \\ \frac{d}{dS}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} &= 0 \\ \frac{d}{dS}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} &= 0 \end{aligned} \quad (31)$$

i.e.

$$
\begin{aligned}
&0=\frac{d}{dS}(c^2\,\dot{t}'+2u\gamma^2\sin\alpha\,\dot{x}'-2u\gamma^2\cos\alpha\,\dot{y}')\\
&0=\frac{d}{dS}(-\dot{x}'+2u\gamma^2\sin\alpha\,\dot{t}')\\
&0=\frac{d}{dS}(\dot{y}'+2u\gamma^2\cos\alpha\,\dot{t}')\\
&\ddot{z}'=0
\end{aligned}
\qquad (32)
$$

In other words:

$$
\begin{aligned}
&c^2\,\dot{t}'+(2u\gamma^2\sin\alpha)\dot{x}'-(2u\gamma^2\cos\alpha)\,\dot{y}'=k\\
&-\dot{x}'+(2u\gamma^2\sin\alpha)\,\dot{t}'=m\\
&\dot{y}'+(2u\gamma^2\cos\alpha)\,\dot{t}'=n\\
&\dot{z}'=p
\end{aligned}
\qquad (33)
$$

The velocity components in the rotating frame are, as a function of the proper time parameter τ given in (34).

$$
\begin{aligned}
&\dot{z}'=p\\
&\dot{x}'=\frac{2ku\gamma^2\sin\alpha-mc^2+2u^2\gamma^4\sin 2\alpha(n-m\cot\alpha)}{c^2-4u^2\gamma^4\cos 2\alpha}\\
&\dot{y}'=\frac{nc^2-4nu^2\gamma^4(\cos 2\alpha+\cos\alpha)+8mu^2\gamma^4\cos 2\alpha\cot\alpha-2ku\gamma^2\cos\alpha}{c^2-4u^2\gamma^4\cos 2\alpha}
\end{aligned}
\qquad (34)
$$

While integrating (34) in order to get the trajectories may prove a difficult task, we have managed to get the symbolic expressions of the velocity components.

IV. Conclusion

We have derived the Euler-Lagrange equations of motion for two main classes of rotating frames of reference: centrally rotating and peripherally rotating frames. We have attempted to "lift the veil" off the more mysterious rotating frames.

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Adrian Sfarti is the author of 32 patents and over 90 refereed papers in the fields of physics, computer science and computer architecture.